

Localization properties of the anomalous diffusion phase in the directed trap model and in the Sinai diffusion with a bias

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We study the localization properties of the anomalous diffusion phase $x \sim t^\mu$ with $0 < \mu < 1$, which exists both in the Sinai diffusion with a small bias, and in the related directed trap model presenting a broad distribution of trapping times $p(\tau) \sim 1/\tau^{1+\mu}$. Our starting point is the real space renormalization method, in which the whole thermal packet is considered to be in the same renormalized valley at large time: this assumption is asymptotically exact only in the limit of vanishing bias $\mu \rightarrow 0$ and corresponds to the Golosov localization. For finite μ , we thus generalize the usual real space renormalization method to allow for the spreading of the thermal packet over many renormalized valleys. Our construction allows one to compute exact series expansions in μ for all observables: to compute observables at order μ^n , it is sufficient to consider in each sample a spreading of the thermal packet onto at most $(1+n)$ traps. So our approach provides a description of the structure of the thermal packet sample by sample, and a full statistical characterization of the important traps at a given order in μ . For the directed trap model, we show explicitly up to order μ^2 how to recover the exact expressions for the diffusion front, the thermal width, and the localization parameter Y_2 . We then use our method to derive exact results for the localization parameters Y_k for arbitrary k , the correlation function of two particles, and the generating function of thermal cumulants. We then explain how these results apply to the Sinai diffusion with bias by deriving the quantitative mapping between the large-scale renormalized descriptions of the two models. Finally we study the internal structure of the effective “traps” for the Sinai model via path-integral methods.

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I. INTRODUCTION

The motivation to study the Sinai model [1] has two origins. On one hand, the Sinai model represents a “toy” disordered system, in which many properties that exist in more complex systems can be studied exactly, such as aging behavior [2,3], persistence exponents [2,4], the decoupling of the dynamics into fast degrees of freedom, which rapidly reach local equilibrium and a slow nonequilibrium part governed by metastable states [5], and some chaos and rejuvenation effects [6]. On the other hand, the Sinai model directly appears in various specific systems ranging from the dynamics of domain walls in the random field Ising chain [7,8] to the unzipping transition in DNA [9]. It is thus interesting to obtain exact detailed information for various observables in the Sinai model.

One of the most important properties of the symmetric Sinai diffusion is the following localization phenomenon discovered by Golosov [10]: all the thermal trajectories starting from the same initial condition in the same sample remain within a finite distance of each other even in the limit of infinite time. In particular, in a given sample, for a given initial condition, the rescaled position $X = x(t)/(\ln t)^2$ is *deterministic*, and it is only after averaging over the samples that X is distributed with the Kesten distribution [2,10,11]. The physical picture is that the particle is at time t near the bottom of the deepest valley it has been able to reach. This is why the real space renormalization group method, first introduced in the field of random quantum spin chains [12,13], is so well suited to study the symmetric Sinai diffusion [2]. Recently [5], we have studied in more detail the localization

properties by computing the infinite-time limit of the localization parameters, which represent the disorder averages of the probabilities that k independent particles in the same sample starting from the same initial condition are at the same place at time t and of the correlation function $C(l, t)$, which represents the disorder average of the probability that two independent particles in the same sample starting from the same initial condition are at a distance l from each other at time t . We have moreover shown [5] that the infinite-time limit of the localization parameters and of the correlation function exactly coincide with the corresponding equilibrium observables in a Brownian potential in the thermodynamic limit.

A natural question is thus: do some of these localization properties survive in the presence of a small bias?

A. Sinai model with bias

The Sinai model in the presence of a constant bias $F_0 > 0$ can be studied in a continuum version via the following Langevin equation [14]:

$$\frac{dx}{dt} = F_0 - U'(x(t)) + \eta(t), \quad (1)$$

where $\eta(t)$ is the usual thermal noise

$$\langle \eta(t) \eta(t') \rangle = 2T \delta(t - t'), \quad (2)$$

and where $U(x)$ is a Brownian random potential representing the disordered landscape,

$$\overline{[U(x) - U(y)]^2} = 2\sigma \delta(x - y). \quad (3)$$

Equivalently, the model may be defined by the Fokker-Planck equation in a given sample $\{U(x)\}$ for the probability distribution $P(x,t|x_0,0)$,

$$\partial_t P(x,t|x_0,0) = \partial_x [T \partial_x + U'(x) - F_0] P(x,t|x_0,0). \quad (4)$$

In the biased case $F_0 > 0$, the diffusion becomes transient, and there are dynamic phase transitions [14–16] as F_0 grows, in terms of the dimensionless parameter

$$\mu = \frac{F_0 T}{\sigma}. \quad (5)$$

For $0 < \mu < 1$, the mean position of the particle presents the anomalous behavior [14–16]

$$\overline{\langle x(t) \rangle} \underset{t \rightarrow \infty}{\propto} t^\mu, \quad (6)$$

whereas for $\mu > 1$, there is a finite velocity $\overline{\langle x(t) \rangle} \sim V(\mu)t$. For the anomalous diffusion phase, the exact diffusion front is given in terms of Lévy stable distributions [14,15,17,18]: we refer the reader to Appendix for the definition and properties of these Lévy fronts.

B. Directed trap model

It has been suggested in Ref. [14] that at large time, the physics of the Sinai model with bias is actually equivalent to a simple directed trap model defined by the master equation [19]

$$\frac{dP_t(n)}{dt} = -\frac{P_t(n)}{\tau_n} + \frac{P_t(n-1)}{\tau_{n-1}}, \quad (7)$$

with the initial condition $P_{t=0}(n) = \delta_{n,0}$, and where the trapping times are independent random variables distributed with a law presenting the algebraic decay

$$q(\tau) \underset{\tau \rightarrow \infty}{\propto} \frac{1}{\tau^{1+\mu}}. \quad (8)$$

So here the random environment consists of the realization of the trapping times $\{\tau_0, \tau_1, \tau_2, \dots\}$. Given this random environment, a directed random walk is defined by the sequence of the sojourn times (t_0, t_1, \dots) , where the sojourn type t_i is a random variable distributed with an exponential distribution of mean τ_i ,

$$f_{\tau_i}(t_i) = \frac{1}{\tau_i} e^{-t_i/\tau_i}. \quad (9)$$

The anomalous diffusion phase $0 < \mu < 1$ then corresponds to the phase where the mean trapping time $\langle \tau \rangle = \int d\tau \tau q(\tau)$ is infinite. The corresponding diffusion front is also a Lévy diffusion front (see Appendix A) as for the biased Sinai diffusion discussed above. For simplicity in this paper, we choose the normalization of the algebraic tail to be

$$q(\tau) \underset{\tau \rightarrow \infty}{\simeq} \frac{\mu}{\tau^{1+\mu}}. \quad (10)$$

It is clear that this choice simply amounts to a rescaling of τ .

The presence of the large algebraic decay in the effective trapping-time distribution (8) for the biased Sinai diffusion may be understood from the real space renormalization approach in relation with the distribution of the barriers against the drift in the renormalized landscape at scale Γ [2]:

$$P_\Gamma(F) = \theta(F - \Gamma) 2 \delta e^{-2\delta(F-\Gamma)}, \quad (11)$$

where $2\delta = F_0/\sigma$. The trapping time $\tau \sim e^{\beta F}$ is then distributed with the power law (8) with the correspondence $\mu = 2\delta T$.

C. Previous results for the localization in the directed trap model

For the directed trap model, the existing results on the extension of the thermal packet are twofold. On one hand, the thermal width has been exactly computed in Ref. [19] [Eq. (26)],

$$\begin{aligned} \overline{\langle \Delta n^2(t) \rangle} &\equiv \overline{\sum_{n=0}^{+\infty} n^2 P_t(n) - \left[\sum_{n=0}^{+\infty} n P_t(n) \right]^2} \\ &= \frac{1}{\Gamma(2\mu)} \left(\frac{\sin \pi \mu}{\pi \mu} \right)^3 I(\mu) t^{2\mu}, \end{aligned} \quad (12)$$

where the integral $I(\mu)$ of Eq. (26) in Ref. [19] can be rewritten after a change of variables as

$$I(\mu) = \int_0^1 dz \frac{(1+z)z^\mu(1-z)^{2\mu}}{z^{2\mu+2} + 2 \cos \pi \mu z^{\mu+1} + 1}. \quad (13)$$

The result (12) shows that the the thermal packet is spread over a length of order t^μ .

On the other hand, the infinite-time limit of the localization parameter for $k=2$ has been exactly computed in Ref. [20]: their result (24) may be rewritten after a deformation of the contour in the complex plane as

$$Y_2(\mu) \equiv \lim_{t \rightarrow \infty} \sum_{n=0}^{+\infty} \overline{[P_t(n)]^2} = \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} \frac{e^{i\theta\mu} - e^{i\theta}}{1 - e^{i\theta(\mu+1)}}. \quad (14)$$

This expression shows that Y_2 is finite in the full domain $0 \leq \mu < 1$ and vanishes in the limit $\mu = 1$. How can this property coexist with the result (12) for the thermal width? The numerical simulations of Ref. [20] show that for a single sample at fixed t , the probability distribution $P_t(n)$ is made out of a few sharp peaks that have a finite weight but that are at a distance of order t^μ . This explains why at the same time, there is a finite probability to find two particles at the same site even at infinite time, even if the thermal width diverges as $t^{2\mu}$ at large time.

D. Goal and results

The aim of this paper is to provide a probabilistic description, sample by sample, of the localization properties of the directed trap model and of the Sinai diffusion with bias in the anomalous diffusion phase $0 < \mu < 1$. More explicitly, the question we address is the following: in a given sample representing a particular realization of the random environment, what is the probability distribution for the position of the random walker?

We will need to generalize the usual real space renormalization group (RSRG) method [2] to allow the spreading of the thermal packet over many renormalized valleys. Indeed, in the usual RSRG method, the whole thermal packet is considered to be in the same renormalized valley at a large time: this assumption is asymptotically exact in the symmetric Sinai model and actually corresponds to the Golosov localization [5,10] discussed above; it is also valid for the biased case but only in the double limit of vanishing bias $\mu \rightarrow 0$ and large time with the fixed parameter $\gamma = \mu T \ln t$ [2]. We will thus define explicit rules for the RSRG approach with multiple valley occupancies and show that our construction allows one to compute exact expansions in μ for all observables.

1. Summary of results for the directed trap model

For the directed trap model, we explicitly show how to recover in a unified framework the expansions up to order μ^2 of the exact results for the observables discussed above.

(a) Expansion in μ of the Lévy diffusion front for the rescaled variable $X = x/t^\mu$ (see Appendix A),

$$g(X) = e^{-X} + \mu \gamma_E (X-1) e^{-X} + \mu^2 \left[\left(\frac{\gamma_E^2}{2} + \frac{\pi^2}{12} \right) + X \left(\frac{\pi^2}{12} - 3 \frac{\gamma_E^2}{2} \right) + X^2 \left(\frac{\gamma_E^2}{2} - \frac{\pi^2}{12} \right) \right] e^{-X} + O(\mu^3). \quad (15)$$

(b) Expansion in μ of the thermal width [from Eqs. (12) and (13)]

$$\Delta(\mu) \equiv \lim_{t \rightarrow \infty} \frac{\overline{\langle \Delta n^2(t) \rangle}}{t^{2\mu}} = \mu(2 \ln 2) + \mu^2 \left[-\frac{\pi^2}{6} + 2 \ln 2 (\ln 2 - 2 + 2 \gamma_E) \right] + O(\mu^3). \quad (16)$$

(c) Expansion in μ of the localization parameter Y_2 [from Eq. (14)],

$$Y_2(\mu) = 1 - \mu(2 \ln 2) + \mu^2 \left(4 \ln 2 - \frac{\pi^2}{6} \right) + O(\mu^3). \quad (17)$$

These comparisons with exact results show that our generalized RSRG procedure is exact order by order in μ : to compute observables at order μ^n , it is sufficient to consider a

spreading of the thermal packet onto at most $(1+n)$ traps. So our description provides a description of the structure of the thermal packet sample by sample, and a full statistical characterization of the important traps at a given order in μ .

We then use our procedure to derive other exact results. We obtain the expansion in μ of the localization parameter Y_k for arbitrary k up to order μ^2

$$Y_k(\mu) = 1 + \mu \int_0^{+\infty} \frac{dv}{v} [e^{-kv} + (1 - e^{-v})^k - 1] + \mu^2 \int_0^{+\infty} \frac{dv}{v} \int_v^{+\infty} \frac{dw}{w} [p_2^k(v, w) + p_2^k(w, v) + 2p_3^k(v, w) + 1 - e^{-kv} - 2(1 - e^{-v})^k - (1 - e^{-w})^k] + O(\mu^3), \quad (18)$$

where the functions p_2 and p_3 are defined in Eq. (89).

We obtain that the correlation function of two particles averages over the disorder reads

$$C(l, t) \equiv \overline{\sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} P(n, t|0, 0) P(m, t|0, 0) \delta_{l, |n-m|}} \simeq_{t \rightarrow \infty} Y_2(\mu) \delta_{l, 0} + \frac{1}{t^\mu} C_\mu \left(\frac{l}{t^\mu} \right), \quad (19)$$

where the weight of the δ peak at the origin corresponds as it should to the localization parameter Y_2 (17), whereas the second part presents a scaling form of the variable $\lambda = l/t^\mu$. We obtain the following expansion for the scaling function C_μ :

$$C_\mu(\lambda) = e^{-\lambda} \left(\mu(2 \ln 2) + \mu^2 \left[\frac{\pi^2}{3} - \ln 2 (4 + \ln 2 + \gamma_E) + \lambda \left(-\frac{\pi^2}{6} + \ln 2 (\ln 2 + \gamma_E) \right) \right] + O(\mu^3) \right). \quad (20)$$

We also consider the generating function of rescaled thermal cumulants $c_k(\mu)$,

$$Z_\mu(s) \equiv \overline{\ln \langle e^{-s(n/t^\mu)} \rangle} = \sum_{k=1}^{+\infty} \frac{(-s)^k}{k!} c_k(\mu). \quad (21)$$

The first one simply represents the mean value that can be obtained from the diffusion front (15)

$$c_1(\mu) = \frac{\langle n \rangle}{t^\mu} = \int_0^{+\infty} dX X f_\mu^{\text{trap}}(X). \quad (22)$$

The second one $c_2(\mu)$ represents the thermal width $\Delta(\mu)$ (16). We obtain the expansion at first order in μ of the generating function

$$Z_\mu(s) = -s + \mu \int_0^{+\infty} dY e^{-Y} \left[\int_0^1 \frac{dv}{v} \ln[e^{-v} + (1 - e^{-v})e^{-sY}] \right. \\ \left. + \int_1^{+\infty} \frac{dv}{v} \ln[e^{-v}e^{sY} + (1 - e^{-v})] \right] + O(\mu^2). \quad (23)$$

The series expansion in s then yields all thermal cumulants at first order in μ . In particular, the first terms beyond the mean value $c_1(\mu)$ and the thermal width $c_2(\mu)$ read

$$c_3(\mu) \equiv \lim_{t \rightarrow \infty} \frac{\overline{\langle n^3 \rangle - 3\langle n^2 \rangle \langle n \rangle + 2\langle n \rangle^3}}{t^{3\mu}} \\ = \mu 6(2 \ln 3 - 3 \ln 2) + O(\mu^2), \quad (24)$$

$$c_4(\mu) \equiv \lim_{t \rightarrow \infty} \frac{\overline{\langle n^4 \rangle - 4\langle n^3 \rangle \langle n \rangle - 3\langle n^2 \rangle^2 + 12\langle n^2 \rangle \langle n \rangle^2 - 6\langle n \rangle^4}}{t^{2\mu}} \\ = \mu 24(19 \ln 2 - 12 \ln 3) + O(\mu^2). \quad (25)$$

2. Summary of results for the biased Sinai model

We will derive an exact quantitative mapping between the renormalized descriptions of the trap model and the biased Sinai diffusion with bias. As a consequence, in the whole anomalous diffusion phase $0 < \mu < 1$, all properties of the directed trap models that concern the rescaled quantity $X = n/t^\mu$ are exactly the same for the Sinai model with bias in terms of the rescaled quantity

$$X = \frac{x\sigma\beta^2}{\Gamma^2(\mu)(t\sigma^2\beta^3)^\mu}. \quad (26)$$

This relation was already conjectured in Ref. [14] for the special case of the averaged diffusion fronts of the two models (see Appendix A). In particular, the thermal width of the Sinai model reads from the exact result (12) of Ref. [19],

$$\frac{\overline{\langle \Delta x^2(t) \rangle}}{t^{2\mu}} = \frac{(\sigma^2\beta^3)^{2\mu}}{\sigma^2\beta^4} \frac{\Gamma^4(\mu)}{\Gamma(2\mu)} \left(\frac{\sin \pi\mu}{\pi\mu} \right)^3 I(\mu) \quad (27) \\ = \frac{(\sigma^2\beta^3)^{2\mu}}{\sigma^2\beta^4} \left\{ \frac{(2 \ln 2)}{\mu^3} + \left[-\frac{\pi^2}{6} \right. \right. \\ \left. \left. + 2 \ln 2 (\ln 2 - 2 - 2\gamma_E) \right] \frac{1}{\mu^2} + O\left(\frac{1}{\mu}\right) \right\}, \quad (28)$$

and more generally, all thermal cumulants can be obtained from the results of the trap model (25) via the correspondence (26).

For the localization parameters, the result Y_k^{trap} represents for the biased Sinai model the probability to find k independent particles at a finite distance from each other in the limit of infinite time. These particles are then distributed with the Boltzmann distribution in an infinitely deep biased Brownian valley, leading to

$$Y_k^{sinai} = Y_k^{trap} Y_k^{valley}, \quad (29)$$

where Y_k^{valley} , computed in Eq. (210), is the localization parameter for k particles at equilibrium in an infinitely deep biased Brownian valley.

For the two-point correlation function, we obtain for the biased Sinai model the two-scaling form

$$C_{sinai}(l, t) = Y_2^{trap} C_{valley}(l) + \frac{\sigma\beta^2}{\Gamma^2(\mu)(t\sigma^2\beta^3)^\mu} C_\mu \\ \times \left(\lambda = \frac{l\sigma\beta^2}{\Gamma^2(\mu)(t\sigma^2\beta^3)^\mu} \right), \quad (30)$$

where the first part represents the case where the two particles are at a finite distance from each other at infinite time, in which case their correlation $C_{valley}(l)$ is given by Eq. (212). The second part, corresponding to the cases where the two particles are in different renormalized valleys at infinite time, is exactly given by the scaling function C_μ (20) describing the long-range behavior in the trap model.

3. Organization of the paper

We first study the directed trap model: Section II presents the usual RSRG that yields all observables in the limit $\mu \rightarrow 0$; in Sec. III, we explain the origin of the spreading of the thermal packet at first order in μ and compute observables at this order; in Sec. IV, we study the second order μ^2 ; in Sec. V, we explain the structure of the set of important traps at any given order μ^n .

We then turn to the biased Sinai model: in Sec. VI, we derive the quantitative mapping between the large-scale renormalized descriptions of the two models (the biased Sinai model and the directed trap model); in Sec. VII, we moreover characterize the internal structure of the ‘‘traps’’ in the biased Sinai model by computing various statistical properties of infinitely biased Brownian valleys. Section VIII contains a discussion on the universality. Finally, Sec. IX contains the conclusion, and some more technical details are given in the Appendixes.

II. DIRECTED TRAP MODEL IN THE LIMIT $\mu \rightarrow 0$

The real space renormalization procedure for the Sinai model [2] can be reformulated for the directed trap model as follows. At time t , all traps with trapping times $\tau_i < t$ are decimated and replaced by a ‘‘flat landscape’’ to produce the renormalized landscape at time t . We stress here that contrary to the symmetric Sinai diffusion, the remaining traps are just some of the initial traps, and that their trapping times have not been renormalized by the decimation of the small ones. This nonrenormalization of the trapping times actually corresponds the biased Sinai landscape to the fact that barriers against the bias converge without rescaling to a fixed distribution [2]. The usual RSRG picture for the dynamics is now very simple: the particle starting at $t=0$ in the $n=0$ trap will be at time t in the first trap of the renormalized landscape, that is, in the first trap having a trapping time bigger than t .

We will call this trap the main trap M . In the usual RSRG approach, all thermal trajectories are in the same trap M . In particular, the probability distribution in a given sample is a δ function,

$$P_t^{(0)}(n) = \delta_{n,n_M} \quad (31)$$

and the localization is total: there are no thermal fluctuations,

$$[\Delta n^2(t)]^{(0)} = 0, \quad (32)$$

and more generally, all thermal cumulants beyond the first one vanish; the generating function of thermal cumulants (21) simply reads

$$Z_\mu^{(0)}(s) = -s \frac{\overline{n_M}}{t^\mu} = -s. \quad (33)$$

The two-particle correlation function is a δ function

$$C^{(0)}(l,t) \equiv \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} P^{(0)}(n) P^{(0)}(m) \delta_{l,|n-m|} = \delta_{l,0} \quad (34)$$

and the localization parameters have their maximal value,

$$Y_k^{(0)}(t) = 1. \quad (35)$$

The corresponding averaged diffusion front is thus simply given by the distribution of the position $n = n_M$ of the main trap

$$\overline{P_t^{(0)}(n)} = \left[1 - \int_t^{+\infty} d\tau q(\tau) \right]^n \int_t^{+\infty} d\tau q(\tau), \quad (36)$$

where the first part $[\dots]^n$ represents the probability that the first n traps have trapping times $\tau_i < t$, and where the last part represents the probability that the n th trap has a trapping time $\tau_i > t$. So the scaling function g describing the averaged diffusion front at large time

$$\overline{P_t^{(0)}(n)} \underset{t \rightarrow \infty}{\simeq} \frac{1}{t^\mu} g\left(\frac{n}{t^\mu}\right) \quad (37)$$

is given at this order by a simple exponential,

$$g^{(0)}(X) = e^{-X}, \quad (38)$$

which indeed coincides with the limit $\mu \rightarrow 0$ of the exact Lévy front (see Appendix A).

So the approximation where all particles of the same thermal packet are in the same trap is correct only in the limit of vanishing μ . For finite μ , we will have to allow for a possible dispersion of the thermal packet. In fact, in the limit $\mu \rightarrow 0$, we have considered that the distribution of the trapping times was infinitely broad in the following sense: all traps with $\tau_i < t$ were such that $\tau_i/t \sim 0$, whereas all traps with $\tau_i > t$ were such that $\tau_i/t \sim +\infty$. For finite μ , we have to take into account that these ratios are not really zero or infinite. We will do it order by order in μ .

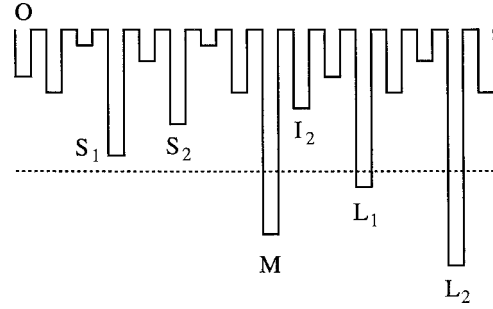


FIG. 1. Hierarchical structure of the important traps for a particle starting at the origin. The dashed line separates the “small” traps (those having a trapping time smaller than t) from the “big” traps (those having a trapping time bigger than t). The first big trap called M is occupied with a weight of order $O(\mu^0)$. The next big trap L_1 and the biggest small trap S_1 before M are occupied with weights of order $O(\mu)$. The third big trap L_2 , the biggest small trap I_2 between M and L_1 , and the second biggest small trap S_2 before M are occupied with weights of order $O(\mu^2)$.

III. DIRECTED TRAP MODEL AT FIRST ORDER IN μ

A. Origins of the dispersion of the thermal packet at order μ

At first order in μ , we need to consider the two following effects (see Fig. 1).

(a) The main trap M defined above has a trapping time τ_M that is not infinite. There is a small probability $(1 - e^{-t/\tau_M})$ that the particle has already escaped from this main trap M at time t to jump into the next renormalized trap that we will call L_1 (for large trap number 1), which is defined as the second trap satisfying $\tau_i > t$.

(b) The biggest trap before the main trap, which we will call S_1 (for small trap number 1), has a trapping time $\tau_{S_1} < t$ that is not zero and thus there is a small probability $e^{-t/\tau_{S_1}}$ that the particle is still trapped in S_1 at time t .

We now describe the statistical properties of these two effects.

B. Statistical properties of the trap L_1

The joint distribution of the trapping time τ_M , of the position $n = n_M$ of the main trap M , and of the position n_L of the next renormalized trap L_1 reads

$$\begin{aligned} \mathcal{D}_{M,L_1}(n, n_L; \tau_M) &= \theta(t < \tau_M) \theta(n < n_L) \\ &\times \left[1 - \int_t^{+\infty} d\tau q(\tau) \right]^n q(\tau_M) \\ &\times \left[1 - \int_t^{+\infty} d\tau q(\tau) \right]^{n_L - n - 1} \int_t^{+\infty} d\tau q(\tau) \\ &\underset{t \rightarrow \infty}{\simeq} \frac{1}{t^{2\mu}} D_{M,L_1} \left(X = \frac{n}{t^\mu}, X_L = \frac{n_L}{t^\mu}; \tau_M \right), \end{aligned} \quad (39)$$

with the scaling function

$$D_{M,L_1}(X, X_L; \tau_M) = \theta(t < \tau_M) \theta(0 < X < X_L) \frac{\mu}{\tau_M} \left(\frac{t}{\tau_M} \right)^\mu e^{-X_L}. \quad (40)$$

In particular, the distribution of the trapping time τ_M is obtained, as it should, by simply normalizing the original distribution $q(\tau)$ on the interval $[t, +\infty]$,

$$q_t(\tau_M) = \int dX \int dX_L D_{M,L_1}(X, X_L; \tau_M) = \theta(\tau_M - t) \frac{\mu t^\mu}{\tau_M^{1+\mu}}. \quad (41)$$

The probability $\pi_{L_1} = (1 - e^{-t/\tau_M})$ to have already escaped from the main trap at time t and to be thus already in the trap L_1 reads after averaging over the disorder,

$$\begin{aligned} \bar{\pi}_{L_1} &= \int d\tau q_t(\tau) (1 - e^{-t/\tau}) = \mu \int_0^1 dv v^{\mu-1} (1 - e^{-v}) \quad (42) \\ &= \mu \int_0^1 \frac{dv}{v} (1 - e^{-v}) + \mu^2 \int_0^1 \frac{dv}{v} \ln v (1 - e^{-v}) + O(\mu^3), \quad (43) \end{aligned}$$

so it is of order μ .

At this level of approximation, the diffusion front for a given sample is made out of two δ distributions,

$$P_{ML_1}^{(0)+(1)}(n) = e^{-t/\tau_M} \delta_{n,n_M} + (1 - e^{-t/\tau_M}) \delta_{n,n_{L_1}}. \quad (44)$$

C. Statistical properties of the trap S_1

The trap S_1 has been defined as the biggest trap before the main trap M . The joint distribution of the position n of the main trap, the position n_S , and the trapping time τ_S of the trap S_1 read

$$\begin{aligned} \mathcal{D}_{S_1,M}(n_S, n; \tau_S) &= \theta(t > \tau_S) \left[1 - \int_{\tau_S}^{+\infty} d\tau q(\tau) \right]^{n-1} \\ &\quad \times q(\tau_S) \int_t^{+\infty} d\tau q(\tau) \quad (45) \\ &\simeq \frac{1}{t^{2\mu}} D_{S_1,M} \left(X_S = \frac{n_S}{t^\mu}, X = \frac{n}{t^\mu}; \tau_S \right), \quad (46) \end{aligned}$$

where the scaling function reads

$$\begin{aligned} D_{S_1,M}(X_S, X; \tau_S) &= \theta(t > \tau_S > 1) \theta(X > X_S > 0) \\ &\quad \times \frac{\mu}{\tau_S} \left(\frac{t}{\tau_S} \right)^\mu e^{-X(t/\tau_S)^\mu}. \quad (47) \end{aligned}$$

We note that here, there are correlations between the trapping time and the positions, contrary to the decoupled measure (40) concerning the trap L_1 . The joint distribution of the positions alone reads

$$\mathcal{S}(X_S, X) = \int d\tau_S D_{S_1,M}(X_S, X; \tau_S) = \frac{\theta(X > X_S > 0)}{X} e^{-X}, \quad (48)$$

i.e., X is distributed with $P^{(0)} = e^{-X}$ (38), and X_S is uniformly distributed on the interval $[0, X]$. The distribution of the trapping time τ_S alone reads

$$\begin{aligned} r(\tau_S) &= \int_0^{+\infty} dX_S \int_{X_S}^{+\infty} dX D_{S_1,M}(X_S, X; \tau_S) \\ &= \theta(t > \tau_S) \mu \frac{\tau_S^{\mu-1}}{t^\mu}. \quad (49) \end{aligned}$$

The probability $\pi_{S_1} = e^{-t/\tau_S}$ to be still in the trap S_1 at time t reads after averaging over the disorder,

$$\begin{aligned} \bar{\pi}_{S_1} &= \int d\tau_S r(\tau_S) e^{-t/\tau_S} = \mu \int_1^{+\infty} \frac{dv}{v^{1+\mu}} e^{-v} \quad (50) \\ &= \mu \int_1^{+\infty} \frac{dv}{v} e^{-v} - \mu^2 \int_1^{+\infty} \frac{dv}{v} e^{-v} + O(\mu^3), \quad (51) \end{aligned}$$

so it is of order μ .

At this level of approximation, the probability distribution reads

$$P_{S_1M}^{(0)+(1)}(n) = e^{-t/\tau_S} \delta_{n,n_{S_1}} + (1 - e^{-t/\tau_S}) \delta_{n,n_M}. \quad (52)$$

We now use the statistical properties of the traps L_1 and S_1 to compute various observables at order μ .

D. Diffusion front at order μ

The correction due to the trap L_1 to the diffusion front in a given sample (44) with respect to one δ function at the zeroth order (31) reads

$$\begin{aligned} P_{ML_1}^{(1)}(n) &\equiv P_{ML_1}^{(0)+(1)}(n) - P^{(0)}(n) \\ &= (1 - e^{-t/\tau_M}) (\delta_{n,n_{L_1}} - \delta_{n,n_M}). \quad (53) \end{aligned}$$

The average over the samples, that is, over the positions (n_M, n_{L_1}) and the trapping time τ with the measure (40) yields the correction to the scaling function (37),

$$\begin{aligned} g_{ML_1}^{(1)}(Y) &= \int d\tau_M \int dX \int dX_L D_{M,L_1}(X, X_L; \tau_M) \\ &\quad \times (1 - e^{-t/\tau_M}) [\delta(Y - X_L) - \delta(Y - X)] \quad (54) \\ &= e^{-X}(X-1) \mu \int_0^1 dv v^{\mu-1} (1 - e^{-v}) \quad (55) \\ &= e^{-X}(X-1) \left[\mu \int_0^1 \frac{dv}{v} (1 - e^{-v}) \right. \\ &\quad \left. + \mu^2 \int_0^1 \frac{dv}{v} \ln v (1 - e^{-v}) + O(\mu^3) \right]. \quad (56) \end{aligned}$$

Similarly, the correction due to the trap S_1 to the diffusion front in a given sample (52) with respect to one δ function at the zeroth order (31) reads

$$P_{S_1M}^{(1)}(n) \equiv P_{S_1M}^{(0)+(1)}(n) - P^{(0)}(n) = e^{-t/\tau_S}(\delta_{n,n_{S_1}} - \delta_{n,n_M}). \quad (57)$$

After averaging over the samples with the measure (47), we obtain the correction to the scaling function (37),

$$g_{S_1M}^{(1)}(Y) = \int d\tau_S \int dX \int dX_S D_{S_1,M}(X_S, X; \tau_S) \times e^{-t/\tau_S}[\delta(Y - X_S) - \delta(Y - X)] \quad (58)$$

$$= \mu \int_1^{+\infty} \frac{dv}{v} e^{-v} (1 - Yv^\mu) e^{-Yv^\mu} \quad (59)$$

$$= \mu e^{-X} (1 - X) \int_1^{+\infty} \frac{dv}{v} e^{-v} + \mu^2 e^{-X} (X^2 - 2X) \times \int_1^{+\infty} \frac{dv}{v} \ln v e^{-v} + O(\mu^3). \quad (60)$$

Adding these two contributions to the zeroth-order front (38), we finally get

$$g_{total}^{(0)+(1)} \equiv g^{(0)}(X) + g_{ML_1}^{(1)}(X) + g_{S_1M}^{(1)}(X) = e^{-X} + e^{-X}(X-1)\mu\gamma_E + O(\mu^2), \quad (61)$$

which coincides with the expansion at order μ of the exact Lévy front (15).

E. Thermal width at order μ

For a given sample, the contribution of the trap L_1 to the thermal width reads, Eq. (44),

$$[\langle \Delta n^2(t) \rangle]_{ML_1}^{(1)} = \overline{\langle n^2 \rangle - \langle n \rangle^2} = e^{-t/\tau_M} (1 - e^{-t/\tau_M}) (n_L - n_M)^2. \quad (62)$$

Averaging over the disorder, that is, over the positions and the trapping time τ with the measure (40) yields

$$[\Delta(\mu)]_{ML_1}^{(1)} = \left[\frac{\langle \Delta n^2(t) \rangle}{t^{2\mu}} \right]_{ML_1}^{(1)} = 2\mu \int_0^1 dv v^{\mu-1} e^{-v} (1 - e^{-v}) \quad (63)$$

$$= 2\mu \int_0^1 \frac{dv}{v} e^{-v} (1 - e^{-v}) + 2\mu^2 \int_0^1 \frac{dv}{v} \ln v e^{-v} (1 - e^{-v}) + O(\mu^3). \quad (64)$$

Similarly, the contribution of the trap S_1 (52) averaged over the samples with the measure (47) reads

$$[\Delta(\mu)]_{S_1M}^{(1)} = \left[\frac{\langle \Delta n^2(t) \rangle}{t^{2\mu}} \right]_{S_1M}^{(1)} = e^{-(t/\tau_S)} (1 - e^{-(t/\tau_S)}) \frac{(n_L - n_S)^2}{t^{2\mu}} = 2\mu \int_1^{+\infty} dv v^{-1-3\mu} e^{-v} (1 - e^{-v}) \quad (65)$$

$$= 2\mu \int_1^{+\infty} \frac{dv}{v} e^{-v} (1 - e^{-v}) - 6\mu^2 \int_1^{+\infty} \frac{dv}{v} \ln v e^{-v} (1 - e^{-v}) + O(\mu^3). \quad (66)$$

Adding the two contributions finally yields

$$[\Delta(\mu)]_{total}^{(1)} = 2\mu \int_0^{+\infty} \frac{dv}{v} e^{-v} (1 - e^{-v}) = \mu(2 \ln 2) \quad (67)$$

in agreement with the exact result (16) of Ref. [19].

F. Localization parameters at order μ

For a given sample, the contribution of the trap L_1 (44) to the localization parameter Y_k representing the probability to find k independent particles in the same trap at time t reads

$$[Y_k]_{ML_1}^{(0)+(1)}(t) = (e^{-t/\tau_M})^k + (1 - e^{-t/\tau_M})^k. \quad (68)$$

So after averaging over the samples, that is, over the trapping time τ (41), the correction to the zeroth order (35) due to the trap L_1 reads

$$[Y_k]_{ML_1}^{(1)}(t) \equiv \overline{[Y_k]_{ML_1}^{(0)+(1)}(t) - Y_k^{(0)}(t)} = \mu \int_0^1 dv v^{\mu-1} [e^{-kv} + (1 - e^{-v})^k - 1]. \quad (69)$$

Similarly, the correction to the zeroth order (35) due to the trap S_1 reads after averaging over the samples, that is, over the trapping time τ_S (49),

$$[Y_k]_{S_1M}^{(1)}(t) \equiv \overline{[Y_k]_{S_1M}^{(0)+(1)}(t) - Y_k^{(0)}(t)} = \mu \int_1^{+\infty} dv v^{-\mu-1} [e^{-kv} + (1 - e^{-v})^k - 1]. \quad (70)$$

Adding these two contributions, we finally get at first order in μ ,

$$Y_k^{(0)+(1)} \equiv Y_k^{(0)} + [Y_k]_{ML_1}^{(1)}(t) + [Y_k]_{S_1M}^{(1)}(t) = 1 + \mu \int_0^{+\infty} \frac{dv}{v} [e^{-kv} + (1 - e^{-v})^k - 1] + O(\mu^2). \quad (71)$$

For the special case $k=2$, the result

$$\overline{Y_2^{(0)+(1)}} = 1 - (2 \ln 2)\mu + O(\mu^2) \quad (72)$$

is again in agreement with the expansion (14) of the exact result [20]. The other first values of k yield

$$\overline{Y_3^{(0)+(1)}} = 1 - (3 \ln 2)\mu + O(\mu^2), \quad (73)$$

$$\overline{Y_4^{(0)+(1)}} = 1 - \left(2 \ln \frac{32}{9}\right)\mu + O(\mu^2). \quad (74)$$

G. Correlation function of two particles at order μ

We now consider the correlation function of two particles (19). For a given sample, the contribution of the trap L_1 (44) reads

$$C_{ML_1}^{(0)+(1)}(l,t) \equiv \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} P_{ML_1}^{(0)+(1)}(n) P_{ML_1}^{(0)+(1)}(m) \delta_{l,|n-m|} \quad (75)$$

$$= [(e^{-t/\tau_M})^2 + (1 - e^{-t/\tau_M})^2] \delta_{l,0} + 2e^{-t/\tau_M} (1 - e^{-t/\tau_M}) \delta_{l,(n_{L_1} - n_M)}. \quad (76)$$

After averaging over the disorder with the measure (40), the

correction with respect to the zeroth order of the correlation function (34) reads

$$\begin{aligned} \overline{C_{ML_1}^{(1)}}(l,t) &\equiv \overline{C_{ML_1}^{(0)+(1)}(l,t) - C^{(0)}(l,t)} \\ &\simeq \overline{[Y_2]_{ML_1}^{(1)}} \delta_{l,0} + 2\mu \frac{1}{t^\mu} e^{-l/t^\mu} \\ &\quad \times \int_0^1 dv v^{\mu-1} e^{-v} (1 - e^{-v}). \end{aligned} \quad (77)$$

It presents the form (19): the weight of the δ part has been obtained in Eq. (69) and the scaling function reads

$$\begin{aligned} [\mathcal{C}_\mu(\lambda)]_{ML_1} &= e^{-\lambda} 2\mu \int_0^1 dv v^{\mu-1} e^{-v} (1 - e^{-v}) \\ &= e^{-\lambda} \left[2\mu \int_0^1 \frac{dv}{v} e^{-v} (1 - e^{-v}) \right. \\ &\quad \left. + 2\mu^2 \int_0^1 \frac{dv}{v} \ln v e^{-v} (1 - e^{-v}) + O(\mu^3) \right]. \end{aligned} \quad (79)$$

Similarly, the contribution of the trap S_1 reads after averaging over the samples with the measure (47),

$$\begin{aligned} \overline{C_{S_1 M}^{(1)}}(l,t) &\equiv \overline{C_{S_1 M}^{(0)+(1)} - C^{(0)}}(l,t) = \overline{[(e^{-t/\tau_S})^2 + (1 - e^{-t/\tau_S})^2 - 1] \delta_{l,0} + 2e^{-t/\tau_S} (1 - e^{-t/\tau_S}) \delta_{l,(n_M - n_{S_1})}} \\ &\simeq \overline{[Y_2]_{S_1 M}^{(1)}} \delta_{l,0} + 2\mu \int_1^{+\infty} \frac{dv}{v} e^{-v} (1 - e^{-v}) \frac{1}{t^\mu} e^{-(l/t^\mu)v^\mu}. \end{aligned} \quad (80)$$

It presents the form (19): the weight of the δ part has been obtained in Eq. (70) and the scaling function reads

$$\begin{aligned} [\mathcal{C}_\mu(\lambda)]_{S_1 M} &= 2\mu \int_1^{+\infty} \frac{dv}{v} e^{-v} (1 - e^{-v}) e^{-\lambda v^\mu} \\ &= e^{-\lambda} \left[2\mu \int_1^{+\infty} \frac{dv}{v} e^{-v} (1 - e^{-v}) \right. \\ &\quad \left. - 2\mu^2 \lambda \int_1^{+\infty} \frac{dv}{v} \ln v e^{-v} (1 - e^{-v}) \right]. \end{aligned} \quad (82)$$

Adding these two contributions, we finally get at first order in μ the following correlation function:

$$\begin{aligned} \overline{C^{(0)+(1)}}(l,t) &= \overline{C^{(0)}}(l,t) + \overline{C_{ML_1}^{(1)}} + \overline{C_{S_1 M}^{(1)}} \\ &\simeq [1 - (2 \ln 2)\mu + O(\mu^2)] \delta_{l,0} + \frac{1}{t^\mu} e^{-l/t^\mu} \\ &\quad \times [(2 \ln 2)\mu + O(\mu^2)]. \end{aligned} \quad (83)$$

H. Generating function of thermal cumulants at first order in μ

The correction to the generating function (21) due to the trap L_1 (44) with respect to the zeroth order (33) reads with the measure (40),

$$\begin{aligned} [Z_\mu(s)]_{ML_1}^{(1)} &\equiv [Z_\mu(s)]_{ML_1}^{(0)+(1)} - Z_\mu^{(0)}(s) \\ &= \overline{\ln[e^{-t/\tau_M} + (1 - e^{-t/\tau_M}) e^{-s(X_L - X)}]} \\ &= \mu \int_0^1 dv v^{\mu-1} \int_0^{+\infty} dY e^{-Y} \\ &\quad \times \ln[e^{-v} + (1 - e^{-v}) e^{-sY}]. \end{aligned} \quad (84)$$

Similarly, the correction due to the trap S_1 reads with the measure (47),

$$\begin{aligned}
 [Z_\mu(s)]_{S_1M}^{(1)} &\equiv [Z_\mu(s)]_{S_1M}^{(0)+(1)} - Z_\mu^{(0)}(s) \\
 &= \overline{\ln[e^{-t/\tau_s} e^{s(X-X_s)} + (1 - e^{-t/\tau_s})]} \\
 &= \mu \int_1^{+\infty} \frac{dv}{v} \int_0^{+\infty} dY e^{-Yv^\mu} \\
 &\quad \times \ln[e^{-v} e^{sY} + (1 - e^{-v})]. \quad (85)
 \end{aligned}$$

The total correction at order μ thus reads

$$\begin{aligned}
 [Z_\mu(s)]_{total}^{(1)} &= [Z_\mu(s)]_{ML_1}^{(1)} + [Z_\mu(s)]_{S_1M}^{(1)} \\
 &= \mu \int_0^{+\infty} dY e^{-Y} \left[\int_0^1 \frac{dv}{v} \ln[e^{-v} + (1 - e^{-v}) e^{-sY}] \right. \\
 &\quad \left. + \int_1^{+\infty} \frac{dv}{v} \ln[e^{-v} e^{sY} + (1 - e^{-v})] \right]. \quad (86)
 \end{aligned}$$

We may now perform a series expansion in s and evaluate the integrals to obtain the generating function of all thermal cumulants at first order in μ ,

$$\begin{aligned}
 [Z_\mu(s)]_{total}^{(1)} &= -s \gamma_E + s^2 \ln 2 - s^3 (2 \ln 3 - 3 \ln 2) \\
 &\quad + s^4 (19 \ln 2 - 12 \ln 3) + O(s^5) \quad (87)
 \end{aligned}$$

leading to the results (25).

IV. DIRECTED TRAP MODEL AT ORDER μ^2

A. Dispersion of the thermal packet at order μ^2

To compute observables at order μ^2 , we now have to consider the possible dispersions of the thermal packet over three traps. Denoting by τ_1 and τ_2 the first two trapping times, the occupation probabilities of the three ordered sites are given by

$$\begin{aligned}
 p_1(t; \tau_1) &= e^{-t/\tau_1}, \\
 p_2(t; \tau_1, \tau_2) &= \frac{\tau_2}{\tau_2 - \tau_1} (e^{-t/\tau_2} - e^{-t/\tau_1}), \\
 p_3(t; \tau_1, \tau_2) &= 1 - \frac{\tau_2 e^{-t/\tau_2} - \tau_1 e^{-t/\tau_1}}{\tau_2 - \tau_1}. \quad (88)
 \end{aligned}$$

In the following, we will also use the notations $v = t/\tau_1$, $w = t/\tau_2$:

$$\begin{aligned}
 p_1(v) &= e^{-v}, \\
 p_2(v, w) &= \frac{v}{v - w} (e^{-w} - e^{-v}), \\
 p_3(v, w) &= 1 - p_1(v) - p_2(v, w). \quad (89)
 \end{aligned}$$

To simplify computations later, it will be convenient to use in intermediate calculations the two following obvious properties: the occupation probability of the third site is a symmetric function of (v, w) ,

$$p_3(v, w) = p_3(w, v); \quad (90)$$

and the three occupation probabilities satisfy the normalization

$$p_1(v) + p_2(v, w) + p_3(v, w) = 1. \quad (91)$$

The enumeration of the various possibilities for the three traps is as follows (see Fig. 1).

1. Configurations (M, L_1, L_2)

The three traps are the main trap M , the next renormalized trap L_1 , introduced in Sec. III B, and the second next renormalized trap that we call L_2 . The joint distribution of the rescaled positions and trapping times read

$$\begin{aligned}
 T_{M, L_1, L_2}(X, X_{ML_1}, X_{L_2}; \tau_M, \tau_{L_1}) \\
 &= \theta(t < \tau_M) \theta(t < \tau_{L_1}) \theta(0 \leq X \leq X_1 \leq X_2) \\
 &\quad \times \frac{\mu}{\tau_M} \left(\frac{t}{\tau_M} \right)^\mu \frac{\mu}{\tau_{L_1}} \left(\frac{t}{\tau_{L_1}} \right)^\mu e^{-X_2}. \quad (92)
 \end{aligned}$$

At this level of approximation, the diffusion front is made out of three δ peaks as

$$\begin{aligned}
 P_{ML_1L_2}^{(0)+(1)+(2)}(n) &= p_1(t; \tau_M) \delta_{m, n_M} + p_2(t; \tau_M, \tau_{L_1}) \\
 &\quad \times \delta_{m, n_{L_1}} + p_3(t; \tau_M, \tau_{L_1}) \delta_{m, n_{L_2}}, \quad (93)
 \end{aligned}$$

where the weights of the three traps are given by Eq. (88).

2. Configurations (M, I_2, L_1)

The tree traps are the main trap M , the next renormalized trap L_1 , introduced in Sec. III B, and in between the intermediate trap that we call I_2 , defined as the biggest trap in the decimated region between M and L_1 . The joint distribution of the rescaled positions and trapping times read

$$\begin{aligned}
 T_{M, I_2, L_1}(X, X_I, X_L; \tau_M, \tau_I) \\
 &= \theta(\tau_M > t > \tau_I > 1) \theta(X_L > X_I > X > 0) \\
 &\quad \times \frac{\mu}{\tau_M} \left(\frac{t}{\tau_M} \right)^\mu \frac{\mu}{\tau_I} \left(\frac{t}{\tau_I} \right)^\mu e^{-X} e^{-(X_L - X)(t/\tau_I)^\mu}. \quad (94)
 \end{aligned}$$

The corresponding diffusion front reads

$$\begin{aligned}
 P_{MI_2L_1}^{(0)+(1)+(2)}(n) &= p_1(t; \tau_M) \delta_{m, n_M} + p_2(t; \tau_M, \tau_I) \delta_{m, n_{I_2}} \\
 &\quad + p_3(t; \tau_M, \tau_I) \delta_{m, n_{L_1}}, \quad (95)
 \end{aligned}$$

where the weights are given by Eq. (88).

3. Configurations (S_1, S_2, M) and (S'_2, S_1, M)

The tree traps are the main trap M , the trap S_1 defined before as the biggest trap before M , and the second biggest trap before M , which we call S_2 if its position is between S_1 and M , and S'_2 if its position is between 0 and S_1 .

For the configurations (S_1, S_2, M) , the joint distribution of the rescaled positions and trapping times is given by

$$\begin{aligned} T_{S_1, S_2, M}(X_1, X_2, X; \tau_{S_1}, \tau_{S_2}) \\ = \theta(\tau_{S_2} < \tau_{S_1} < t) \theta(0 < X_1 < X_2 < X) \\ \times \frac{\mu}{\tau_{S_1}} \left(\frac{t}{\tau_{S_1}} \right)^\mu \frac{\mu}{\tau_{S_2}} \left(\frac{t}{\tau_{S_2}} \right)^\mu e^{-X(t/\tau_{S_2})^\mu} \end{aligned} \quad (96)$$

and the corresponding diffusion front reads

$$\begin{aligned} P_{S_1 S_2 M}^{(0)+(1)+(2)}(n) = p_1(t; \tau_{S_1}) \delta_{m, n_{S_1}} + p_2(t; \tau_{S_1}, \tau_{S_2}) \delta_{m, n_{S_2}} \\ + p_3(t; \tau_{S_1}, \tau_{S_2}) \delta_{m, n_M}, \end{aligned} \quad (97)$$

where the weights are given by Eq. (88).

For the configurations (S_2', S_1, M) , the joint distribution of the rescaled positions and trapping times read

$$\begin{aligned} T_{S_2', S_1, M}(X_2, X_1, X; \tau_{S_2}, \tau_{S_1}) \\ = \theta(\tau_{S_2} < \tau_{S_1} < t) \theta(0 < X_2 < X_1 < X) \\ \times \frac{\mu}{\tau_{S_1}} \left(\frac{t}{\tau_{S_1}} \right)^\mu \frac{\mu}{\tau_{S_2}} \left(\frac{t}{\tau_{S_2}} \right)^\mu e^{-X(t/\tau_{S_2})^\mu} \end{aligned} \quad (98)$$

and the corresponding diffusion front reads

$$\begin{aligned} P_{S_2' S_1 M}^{(0)+(1)+(2)}(n) = p_1(t; \tau_{S_2}) \delta_{m, n_{S_2'}} + p_2(t; \tau_{S_2}, \tau_{S_1}) \delta_{m, n_{S_1}} \\ + p_3(t; \tau_{S_2}, \tau_{S_1}) \delta_{m, n_M}, \end{aligned} \quad (99)$$

where the weights are given by Eq. (88).

4. Configurations (S_1, M, L_1)

The three traps are the trap S_1 introduced in Sec. III C, the main trap M , and the next renormalized trap L_1 introduced in Sec. III B. The joint distribution of the rescaled positions and trapping times is given by

$$\begin{aligned} T_{S_1, M, L_1}(X_S, X, X_L; \tau_S, \tau_M) \\ = \theta(\tau_M > t > \tau_S > 1) \theta(X_L > X > X_S > 0) \\ \times \frac{\mu}{\tau_S} \left(\frac{t}{\tau_S} \right)^\mu \frac{\mu}{\tau_M} \left(\frac{t}{\tau_M} \right)^\mu e^{-X(t/\tau_S)^\mu} e^{-(X_L - X)}. \end{aligned} \quad (100)$$

The corresponding diffusion front reads

$$\begin{aligned} P_{S_1 M L_1}^{(0)+(1)+(2)}(n) = p_1(t; \tau_{S_1}) \delta_{m, n_{S_1}} + p_2(t; \tau_{S_1}, \tau_M) \delta_{m, n_M} \\ + p_3(t; \tau_{S_1}, \tau_M) \delta_{m, n_{L_1}}, \end{aligned} \quad (101)$$

where the weights are given by Eq. (88).

We now use the statistical properties of these three-trap configurations to compute observables at order μ^2 .

B. Diffusion front at order μ^2

1. Contributions at order μ^2 of the two-trap configurations

We have already studied the contributions of two-trap configurations when studying the order μ . The contribution of order μ^2 of the configurations ML_1 (56) reads

$$g_{ML_1}^{(2)}(X) = \mu^2 e^{-X} (X-1) \int_0^1 \frac{dv}{v} \ln v (1 - e^{-v}). \quad (102)$$

Similarly, the contribution of order μ^2 of the configurations $S_1 M$ (60) reads

$$g_{S_1 M}^{(2)}(X) = \mu^2 e^{-X} (X^2 - 2X) \int_1^{+\infty} \frac{dv}{v} \ln v e^{-v}. \quad (103)$$

2. Contributions at order μ^2 of the three-trap configurations

The specific contribution at order μ^2 of the three-trap configurations of type $ML_1 L_2$ can be obtained by subtracting from Eq. (93) the two-trap configurations ML_1 (44):

$$\begin{aligned} P_{ML_1 L_2}^{(2)}(n) &\equiv P_{ML_1 L_2}^{(0)+(1)+(2)}(n) - P_{ML_1}^{(0)+(1)+(2)}(n) \\ &= p_3(t; \tau_M, \tau_{L_1}) (\delta_{m, n_{L_2}} - \delta_{m, n_{L_1}}). \end{aligned} \quad (104)$$

The average over the samples with the measure (92) yields the correction of the scaling function (37),

$$\begin{aligned} g_{ML_1 L_2}^{(2)}(Y) &= \int dX \int dX_1 \int dX_2 \int \tau_M \int \tau_{L_1} \\ &\times T_{M, L_1, L_2}(X, X_{ML_1}, X_{L_2}; \tau_M, \tau_{L_1}) \end{aligned} \quad (105)$$

$$p_3(t; \tau_M, \tau_{L_1}) [\delta(Y - X_2) - \delta(Y - X_1)] \quad (106)$$

$$\begin{aligned} &= e^{-Y} \left[\frac{Y^2}{2} - Y \right] \mu^2 \int_0^1 \frac{dv}{v} \\ &\times \int_0^1 \frac{dw}{w} p_3(v, w) + O(\mu^3). \end{aligned} \quad (107)$$

Similarly, the specific contribution at order μ^2 of the three-trap configurations of type $MI_2 L_1$ can be obtained by subtracting from Eq. (95) the two-trap configurations ML_1 (44), and this yields after averaging over the samples with the measure (94),

$$P_{MI_2 L_1}^{(2)}(n) \equiv P_{MI_2 L_1}^{(0)+(1)+(2)}(n) - P_{ML_1}^{(0)+(1)+(2)}(n) = p_2(t; \tau_M, \tau_I) (\delta_{m, n_{I_2}} - \delta_{m, n_{L_1}}). \quad (108)$$

The correction to the scaling function (37) thus reads

$$g_{MI_2L_1}^{(2)}(Y) = e^{-Y} \left[Y - \frac{Y^2}{2} \right] \mu^2 \int_0^1 \frac{dv}{v} \int_1^{+\infty} \frac{dw}{w} p_2(v, w) + O(\mu^3). \quad (109)$$

For the three-trap configurations of type S_1S_2M , we have to subtract from Eq. (97) the two-trap configurations S_1M (52) and to average over the samples with the measure (96)

$$\overline{P_{S_1S_2M}^{(2)}(n)} \equiv \overline{P_{S_1S_2M}^{(0)+(1)+(2)}(n)} - \overline{P_{S_1M}^{(0)+(1)+(2)}(n)} = \overline{p_2(t; \tau_{S_1}, \tau_{S_2}) (\delta_{m, n_{S_2}} - \delta_{m, n_M})}. \quad (110)$$

The correction to the scaling function (37) thus reads

$$g_{S_1S_2M}^{(2)}(Y) = e^{-Y} \left[Y - \frac{Y^2}{2} \right] \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_v^{+\infty} \frac{dw}{w} p_2(v, w) + O(\mu^3). \quad (111)$$

For the three-trap configurations of type $S_2'S_1M$, we have to subtract from Eq. (99) the two-trap configurations S_1M (52) and to average over the samples with the measure (98) as

$$\overline{P_{S_2'S_1M}^{(2)}(n)} \equiv \overline{P_{S_2'S_1M}^{(0)+(1)+(2)}(n)} - \overline{P_{S_1M}^{(0)+(1)+(2)}(n)} \quad (112)$$

$$= \overline{p_1(t; \tau_{S_2}) \delta_{m, n_{S_2'}} + (p_2(t; \tau_{S_2}, \tau_{S_1}) - p_2(t; 0, \tau_{S_1})) \delta_{m, n_{S_1}} + (p_3(t; \tau_{S_2}, \tau_{S_1}) - p_3(t; 0, \tau_{S_1})) \delta_{m, n_M}}. \quad (113)$$

The correction to the scaling function (37) reads

$$g_{S_2'S_1M}^{(2)}(Y) = e^{-Y} (1-Y) \mu^2 \int_1^{+\infty} \frac{dv}{v} \ln v e^{-v} + e^{-Y} \left(Y - \frac{Y^2}{2} \right) \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_1^{+\infty} \frac{dw}{w} \theta(v < w) [p_2(v, w)] + O(\mu^3). \quad (114)$$

For the three-trap configurations of type S_1ML_1 , we have to subtract from Eq. (101) the one-trap configuration (31), and the corrections due to the two-trap configurations ML_1 (53) and S_1M (57), and to average over the samples with the measure (100),

$$\overline{P_{S_1ML_1}^{(2)}(n)} \equiv \overline{P_{S_1ML_1}^{(0)+(1)+(2)}(n)} - \overline{P_M^{(0)}(n)} - \overline{P_{S_1M}^{(1)+(2)}(n)} - \overline{P_{ML_1}^{(1)+(2)}(n)} = \overline{[p_3(t; \tau_{S_1}, \tau_M) - p_3(t; 0, \tau_M)] (\delta_{m, n_{L_1}} - \delta_{m, n_M})}, \quad (115)$$

with the scaling function

$$g_{S_1ML_1}^{(2)}(Y) = e^{-Y} \left[Y - \frac{Y^2}{2} \right] \mu^2 \int_1^{+\infty} \frac{dw}{w} \int_0^1 \frac{dv}{v} p_2(v, w) + O(\mu^3). \quad (116)$$

The sum of all contributions of order μ^2 finally reads

$$g_{\text{total}}^{(2)}(Y) \equiv g_{MI_2L_1}^{(2)}(Y) + g_{S_1ML_1}^{(2)}(Y) + g_{S_1S_2M}^{(2)}(Y) + g_{S_2'S_1M}^{(2)}(Y) + g_{S_1M}^{(2)}(Y) + g_{ML_1L_2}^{(2)}(Y) + g_{ML_1}^{(2)}(Y) \quad (117)$$

$$= e^{-Y} [2Y - Y^2] \mu^2 \int_0^1 \frac{dv}{v} \int_v^{+\infty} \frac{dw}{w} p_2(v, w) + \mu^2 e^{-Y} (Y^2 - 3Y + 1) \left[\int_1^{+\infty} \frac{dv}{v} \ln v e^{-v} - \int_0^1 \frac{dv}{v} \ln v (1 - e^{-v}) \right] + O(\mu^3). \quad (118)$$

The double integral may be computed as follows:

$$\begin{aligned} & \int_0^{+\infty} \frac{dv}{v} \int_0^{+\infty} \frac{dw}{w} p_2(v, w) \\ &= \int_0^{+\infty} \frac{dw}{w} \int_0^1 dz \frac{1}{z-1} (e^{-wz} - e^{-w}) \\ &= \int_0^1 dz \frac{1}{1-z} (\ln z) = -\frac{\pi^2}{6} \end{aligned} \quad (119)$$

and we obtain the final result

$$\begin{aligned} g_{\text{total}}^{(2)}(Y) &= \mu^2 e^{-Y} \left[\left(\frac{\gamma_E^2}{2} + \frac{\pi^2}{12} \right) + Y \left(-3 \frac{\gamma_E^2}{2} + \frac{\pi^2}{12} \right) \right. \\ &\quad \left. + Y^2 \left(\frac{\gamma_E^2}{2} - \frac{\pi^2}{12} \right) \right], \end{aligned} \quad (120)$$

which coincides with the expansion (15) of the exact diffusion front described in Appendix A.

C. Thermal width at order μ^2

1. Contributions at order μ^2 of the two-trap configurations

We have already studied the contributions of two-trap configurations when studying the order μ . The contribution of order μ^2 of the configurations ML_1 reads, Eq. (64),

$$[\Delta(\mu)]_{ML_1}^{(2)} = 2\mu^2 \int_0^1 \frac{dv}{v} \ln v e^{-v} (1 - e^{-v}), \quad (121)$$

whereas the contribution of the configurations S_1M reads, Eq. (66),

$$[\Delta(\mu)]_{S_1M}^{(2)} = -6\mu^2 \int_1^{+\infty} \frac{dv}{v} \ln v e^{-v} (1 - e^{-v}). \quad (122)$$

2. Contributions at order μ^2 of the three-trap configurations

For a given configuration of three traps situated at (n_1, n_2, n_3) with occupation probabilities (p_1, p_2, p_3) , the thermal width reads

$$\begin{aligned} \langle \Delta n^2(t) \rangle &= p_1 p_2 (n_2 - n_1)^2 + p_1 p_3 (n_3 - n_1)^2 \\ &\quad + p_2 p_3 (n_3 - n_2)^2. \end{aligned} \quad (123)$$

Following the procedure described above for the diffusion front, we obtain the specific contributions at order μ^2 of the various configurations as follows.

The configurations of type ML_1L_2 with the measure (92) give

$$[\Delta(\mu)]_{ML_1L_2}^{(2)} \equiv [\Delta(\mu)]_{ML_1L_2}^{(1)+(2)} - [\Delta(\mu)]_{ML_1}^{(1)+(2)} \quad (124)$$

$$= \overline{p_2(t; \tau_M, \tau_{L_1}) p_3(t; \tau_M, \tau_{L_1}) [X_{L_2} - X_{L_1}]^2 + p_1(t; \tau_M) p_3(t; \tau_M, \tau_{L_1}) [(X_{L_2} - X_M)^2 - (X_{L_1} - X_M)^2]} \quad (125)$$

$$= \mu^2 \int_0^1 \frac{dv}{v} \int_0^1 \frac{dw}{w} [2p_2(v, w) p_3(v, w) + 4p_1(v) p_3(v, w)] + O(\mu^3). \quad (126)$$

The configurations of type MI_2L_1 with the measure (94) give

$$[\Delta(\mu)]_{MI_2L_1}^{(2)} \equiv [\Delta(\mu)]_{MI_2L_1}^{(1)+(2)} - [\Delta(\mu)]_{ML_1}^{(1)+(2)} \quad (127)$$

$$= \overline{p_2(t; \tau_M, \tau_{I_2}) p_3(t; \tau_M, \tau_{I_2}) [X_{L_1} - X_{I_2}]^2 + p_1(t; \tau_M) p_2(t; \tau_M, \tau_{I_2}) [(X_{I_2} - X_M)^2 - (X_{L_1} - X_M)^2]} \quad (128)$$

$$= \mu^2 \int_0^1 \frac{dv}{v} \int_1^{+\infty} \frac{dw}{w} [2p_2(v, w) p_3(v, w) - 4p_1(v) p_2(v, w)]. \quad (129)$$

The configurations of type S_1S_2M with the measure (96) give

$$[\Delta(\mu)]_{S_1S_2M}^{(2)} \equiv [\Delta(\mu)]_{S_1S_2M}^{(1)+(2)} - [\Delta(\mu)]_{S_1M}^{(1)+(2)} \quad (130)$$

$$= \overline{p_2(t; \tau_{S_1}, \tau_{S_2}) p_3(t; \tau_{S_1}, \tau_{S_2}) [X_M - X_{S_2}]^2 + p_1(t; \tau_{S_1}) p_2(t; \tau_{S_1}, \tau_{S_2}) [(X_{S_2} - X_{S_1})^2 - (X_M - X_{S_1})^2]} \quad (131)$$

$$= \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_0^{+\infty} \frac{dw}{w} [2p_2(v, w) p_3(v, w) - 4p_1(v) p_2(v, w)]. \quad (132)$$

The configurations of type $S_2'S_1M$ with the measure (98) give

$$[\Delta(\mu)]_{S_2'S_1M}^{(2)} \equiv [\Delta(\mu)]_{S_2'S_1M}^{(1)+(2)} - [\Delta(\mu)]_{S_1M}^{(1)+(2)} \quad (133)$$

$$\begin{aligned} &= \overline{p_1(t; \tau_{S_2}) p_2(t; \tau_{S_2}, \tau_{S_1}) (X_{S_1} - X_{S_2'})^2 + p_1(t; \tau_{S_2}, \tau_{S_1}) p_3(t; \tau_{S_2}, \tau_{S_1}) (X_M - X_{S_2'})^2} \\ &\quad + \overline{[p_2(t; \tau_{S_2}, \tau_{S_1}) p_3(t; \tau_{S_2}, \tau_{S_1}) - p_2(t; 0, \tau_{S_1}) p_3(t; 0, \tau_{S_1})] (X_M - X_{S_1})^2} \end{aligned} \quad (134)$$

$$= \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_1^v \frac{dw}{w} \{2p_1(v)p_2(v, w) + 6p_1(v)p_3(v, w) + 2p_2(v, w)p_3(v, w) - 2p_1(w)[1 - p_1(w)]\}. \quad (135)$$

The configurations of type S_1ML_1 with the measure (100) give

$$[\Delta(\mu)]_{S_1ML_1}^{(2)} \equiv [\Delta(\mu)]_{S_1ML_1}^{(1)+(2)} - [\Delta(\mu)]_{S_1M}^{(1)+(2)} - [\Delta(\mu)]_{ML_1}^{(1)+(2)} \quad (136)$$

$$\begin{aligned} &= \overline{p_1(t; \tau_{S_1}) p_3(t; \tau_{S_1}, \tau_M) [(X_{L_1} - X_{S_1})^2 - (X_M - X_{S_1})^2]} \\ &\quad + \overline{[p_2(t; \tau_{S_1}, \tau_M) p_3(t; \tau_{S_1}, \tau_M) - p_2(t; 0, \tau_M) p_3(t; 0, \tau_M)] (X_{L_1} - X_M)^2} \end{aligned} \quad (137)$$

$$= \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_0^1 \frac{dw}{w} \{2p_2(v, w)p_3(v, w) + 4p_1(v)p_3(v, w) - 2p_1(w)[1 - p_1(w)]\}. \quad (138)$$

Finally, the sum of all contributions at order μ^2 reads

$$\begin{aligned} [\Delta(\mu)]_{total}^{(2)} &\equiv [\Delta(\mu)]_{S_2'S_1M}^{(2)} + [\Delta(\mu)]_{S_1M}^{(2)} + [\Delta(\mu)]_{S_1S_2M}^{(2)} \\ &\quad + [\Delta(\mu)]_{S_1ML_1}^{(2)} + [\Delta(\mu)]_{ML_2L_1}^{(2)} \\ &\quad + [\Delta(\mu)]_{ML_1L_2}^{(2)} + [\Delta(\mu)]_{ML_1}^{(2)} \\ &= -4\mu^2 \int_0^{+\infty} \frac{dv}{v} \ln v e^{-v} (1 - e^{-v}) \quad (139) \end{aligned}$$

$$\begin{aligned} &+ \mu^2 \int_0^{+\infty} \frac{dv}{v} \int_v^{+\infty} \frac{dw}{w} [4p_2(v, w)p_3(v, w) \\ &\quad - 6p_1(v)p_2(v, w) + 2p_1(w)p_3(v, w)] \\ &\quad + O(\mu^3). \end{aligned} \quad (140)$$

The double integral may be computed as in Eq. (119) and yields

$$\begin{aligned} &\int_0^{+\infty} \frac{dv}{v} \int_v^{+\infty} \frac{dw}{w} [4p_2(v, w)p_3(v, w) - 6p_1(v)p_2(v, w) \\ &\quad + 2p_1(w)p_3(v, w)] = -\frac{\pi^2}{6} - 4 \ln 2, \end{aligned} \quad (141)$$

and thus the final result

$$[\Delta(\mu)]_{total}^{(2)} = \mu^2 \left[2 \ln 2 (\ln 2 + 2\gamma_E) - \frac{\pi^2}{6} - 4 \ln 2 \right] \quad (142)$$

coincides with the expansion of the exact result (16).

D. Localization parameters at order μ^2

1. Contributions at order μ^2 of the two-trap configurations

We have already studied the contributions of two-trap configurations when studying the order μ . The contribution of order μ^2 of the configurations ML_1 (69) reads

$$\overline{[Y_k]_{ML_1}^{(2)}} = \mu^2 \int_0^1 \frac{dv}{v} \ln v [e^{-kv} + (1 - e^{-v})^k - 1], \quad (143)$$

whereas the contribution of order μ^2 of the configurations S_1M (70) reads

$$\overline{[Y_k]_{S_1M}^{(2)}} = -\mu^2 \int_1^{+\infty} \frac{dv}{v} \ln v [e^{-kv} + (1 - e^{-v})^k - 1]. \quad (144)$$

2. Contributions at order μ^2 of the three-trap configurations

For a given configuration of three traps with occupation probabilities (88), the localization parameters read in terms of the variables $v \equiv t/\tau_1$ and $w \equiv t/\tau_2$ as

$$Y_k = p_1^k(v) + p_2^k(v, w) + p_3^k(v, w). \quad (145)$$

Following the procedure described above for the diffusion front, we obtain the specific contributions at order μ^2 of the various configurations as follows.

The configurations of type ML_1L_2 with the measure (92) give

$$\begin{aligned} \overline{[Y_k]_{ML_1L_2}^{(2)}} &= \overline{[Y_k]_{ML_1L_2}^{(0)+(1)+(2)}} - \overline{[Y_k]_{ML_1}^{(0)+(1)+(2)}} \\ &= \mu^2 \int_0^1 \frac{dv}{v} \int_0^1 \frac{dw}{w} [p_2^k(v,w) + p_3^k(v,w) \\ &\quad - (1-p_1(v))^k] + O(\mu^3). \end{aligned} \quad (146)$$

The configurations of type $ML_1L_2L_1$ with the measure (94) give

$$\begin{aligned} \overline{[Y_k]_{ML_2L_1}^{(2)}} &= \overline{[Y_k]_{ML_2L_1}^{(0)+(1)+(2)}} - \overline{[Y_k]_{ML_1}^{(0)+(1)+(2)}} \quad (147) \\ &= \mu^2 \int_0^1 \frac{dv}{v} \int_1^{+\infty} \frac{dw}{w} [p_2^k(v,w) + p_3^k(v,w) \\ &\quad - (1-p_1(v))^k] + O(\mu^3). \end{aligned} \quad (148)$$

The configurations of type S_1S_2M with the measure (96) give

$$\begin{aligned} \overline{[Y_k]_{S_1S_2M}^{(2)}} &= \overline{[Y_k]_{S_1S_2M}^{(0)+(1)+(2)}} - \overline{[Y_k]_{S_1M}^{(0)+(1)+(2)}} \\ &= \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_v^{+\infty} \frac{dw}{w} [p_2^k(v,w) + p_3^k(v,w) \\ &\quad - (1-p_1(v))^k] + O(\mu^3). \end{aligned} \quad (149)$$

The configurations of type S'_2S_1M with the measure (98) give

$$\begin{aligned} \overline{[Y_k]_{S'_2S_1M}^{(2)}} &= \overline{[Y_k]_{S'_2S_1M}^{(0)+(1)+(2)}} - \overline{[Y_k]_{S_1M}^{(0)+(1)+(2)}} \\ &= \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_1^v \frac{dw}{w} \theta(v>w) \{p_1^k(v) + p_2^k(v,w) \\ &\quad + p_3^k(v,w) - p_1^k(w) - [1-p_1(w)]^k\} + O(\mu^3). \end{aligned} \quad (150)$$

The configurations of type S_1ML_1 with the measure (92) give

$$\begin{aligned} \overline{[Y_k]_{S_1ML_1}^{(2)}} &= \overline{[Y_k]_{S_1ML_1}^{(0)+(1)+(2)}} - \overline{[Y_k]_{S_1M}^{(0)}} - \overline{[Y_k]_{S_1M}^{(1)+(2)}} - \overline{[Y_k]_{ML_1}^{(1)+(2)}} \\ &= \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_0^1 \frac{dw}{w} \{p_2^k(v,w) + p_3^k(v,w) + 1 \\ &\quad - [1-p_1(v)]^k - p_1^k(w) - [1-p_1(w)]^k\} + O(\mu^3). \end{aligned} \quad (152)$$

The sum of all contributions of order μ^2 thus reads

$$\begin{aligned} \overline{[Y_k]_{total}^{(2)}} &= \overline{[Y_k]_{ML_2L_1}^{(2)}} + \overline{[Y_k]_{S_1ML_1}^{(2)}} + \overline{[Y_k]_{ML_1L_2}^{(2)}} + \overline{[Y_k]_{ML_1}^{(2)}} \\ &\quad + \overline{[Y_k]_{S_1S_2M}^{(2)}} + \overline{[Y_k]_{S'_2S_1M}^{(2)}} + \overline{[Y_k]_{S_1M}^{(2)}} \\ &= \mu^2 \int_0^1 \frac{dv}{v} \int_v^{+\infty} \frac{dw}{w} \{p_2^k(v,w) + p_2^k(w,v) \\ &\quad + 2p_3^k(v,w) + 1 - p_1^k(v) - 2[1-p_1(v)]^k \\ &\quad - [1-p_1(w)]^k\}. \end{aligned} \quad (153)$$

For the special case $k=2$, we find

$$\overline{[Y_2]_{total}^{(2)}} = \mu^2 \left[4 \ln 2 - \frac{\pi^2}{6} \right] \quad (154)$$

in agreement with the expansion of the exact result (17). For the special case $k=4$, we find

$$\overline{[Y_4]_{total}^{(2)}} = \mu^2 \left(-\frac{\pi^2}{6} + 2[21 \ln 2 + \ln 3(\ln 3 - 12)] \right). \quad (155)$$

E. Correlation function at order μ^2

1. Contributions at order μ^2 of the two-trap configurations

We have already studied the contributions of two-trap configurations when studying the order μ . The contribution of order μ^2 of the configurations ML_1 (79) reads

$$[C_\mu(\lambda)]_{ML_1}^{(2)} = e^{-\lambda} 2\mu^2 \int_0^1 \frac{dv}{v} \ln v e^{-v} (1 - e^{-v}), \quad (156)$$

whereas the contribution of order μ^2 of the configurations S_1M (82) reads

$$[C_\mu(\lambda)]_{S_1M}^{(2)} = -2\mu^2 \lambda e^{-\lambda} \int_1^{+\infty} \frac{dv}{v} \ln v e^{-v} (1 - e^{-v}). \quad (157)$$

2. Contributions at order μ^2 of the three-trap configurations

For a given configuration of three traps situated at (n_1, n_2, n_3) with occupation probabilities (p_1, p_2, p_3) , the two-particle correlation functions reads

$$\begin{aligned} C(l, t) &= (p_1^2 + p_2^2 + p_3^2) \delta_{l,0} + 2p_1p_2 \delta_{l, n_2 - n_1} + 2p_1p_3 \delta_{l, n_3 - n_1} \\ &\quad + 2p_2p_3 \delta_{l, n_3 - n_2}. \end{aligned} \quad (158)$$

Since the weight of the δ peak is given by the localization parameter Y_2 that we have already considered above, we will consider in the following only the scaling function $C_\mu(\lambda)$ (19). Following the procedure described above for the diffusion front, we will obtain the specific contributions at order μ^2 .

The configurations of type ML_1L_2 give the specific contribution at order μ^2 as

$$\begin{aligned}
 [\mathcal{C}_\mu(\lambda)]_{ML_1L_2}^{(2)} &= [\mathcal{C}_\mu(\lambda)]_{ML_1L_2}^{(1)+(2)} - [\mathcal{C}_\mu(\lambda)]_{ML_1}^{(1)+(2)} \\
 &= e^{-\lambda} \mu^2 \int_0^1 \frac{dv}{v} \int_0^1 \frac{dw}{w} [2p_2(v,w)p_3(v,w) \\
 &\quad - 2p_1(v)p_3(v,w) + \lambda 2p_1(v)p_3(v,w)] \\
 &\quad + O(\mu^3). \tag{159}
 \end{aligned}$$

The configurations of type MI_2L_1 give the specific contribution at order μ^2 as

$$\begin{aligned}
 [\mathcal{C}_\mu(\lambda)]_{MI_2L_1}^{(2)} &= [\mathcal{C}_\mu(\lambda)]_{MI_2L_1}^{(1)+(2)} - [\mathcal{C}_\mu(\lambda)]_{ML_1}^{(1)+(2)} \\
 &= e^{-\lambda} \mu^2 \int_0^1 \frac{dv}{v} \int_1^{+\infty} \frac{dw}{w} [2p_1(v)p_2(v,w) \\
 &\quad + 2p_2(v,w)p_3(v,w) - 2p_1(v)p_2(v,w)\lambda] \\
 &\quad + O(\mu^3). \tag{160}
 \end{aligned}$$

The configurations of type S_1S_2M give the specific contribution at order μ^2 as

$$\begin{aligned}
 [\mathcal{C}_\mu(\lambda)]_{S_1S_2M}^{(2)} &= [\mathcal{C}_\mu(\lambda)]_{S_1S_2M}^{(1)+(2)} - [\mathcal{C}_\mu(\lambda)]_{S_1M}^{(1)+(2)} \\
 &= e^{-\lambda} \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_v^{+\infty} \frac{dw}{w} [2p_1(v)p_2(v,w) \\
 &\quad + 2p_2(v,w)p_3(v,w) - 2p_1(v)p_2(v,w)\lambda] \\
 &\quad + O(\mu^3). \tag{161}
 \end{aligned}$$

The configurations of type $S_2'S_1M$ give the specific contribution at order μ^2 as

$$\begin{aligned}
 [\mathcal{C}_\mu(\lambda)]_{S_2'S_1M}^{(2)} &= [\mathcal{C}_\mu(\lambda)]_{S_2'S_1M}^{(1)+(2)} - [\mathcal{C}_\mu(\lambda)]_{S_1M}^{(1)+(2)} \\
 &= e^{-\lambda} \mu^2 \int_1^{+\infty} \frac{dv}{v} \int_v^{+\infty} \frac{dw}{w} \{2p_1(w)p_2(w,v) \\
 &\quad + 2p_2(w,v)p_3(v,w) - 2p_1(v)[1-p_1(v)] \\
 &\quad + 2p_1(w)p_3(v,w)\lambda\} + O(\mu^3). \tag{162}
 \end{aligned}$$

The configurations of type S_1ML_1 give the specific contribution at order μ^2 as

$$\begin{aligned}
 [\mathcal{C}_\mu(\lambda)]_{S_1ML_1}^{(2)} &= [\mathcal{C}_\mu(\lambda)]_{S_1ML_1}^{(1)+(2)} - [\mathcal{C}_\mu(\lambda)]_{S_1M}^{(1)+(2)} \\
 &\quad - [\mathcal{C}_\mu(\lambda)]_{ML_1}^{(1)+(2)} \\
 &= \mu^2 e^{-\lambda} \int_0^1 \frac{dv}{v} \int_1^{+\infty} \frac{dw}{w} \{2p_2(w,v)p_3(v,w) \\
 &\quad - 2p_1(w)p_3(v,w) - 2p_1(v)[1-p_1(v)] \\
 &\quad + 2p_1(w)p_3(v,w)\lambda\} + O(\mu^3). \tag{163}
 \end{aligned}$$

The sum of all contributions of order μ^2 reads

$$\begin{aligned}
 [\mathcal{C}_\mu(\lambda)]_{total}^{(2)} &= [\mathcal{C}_\mu(\lambda)]_{MI_2L_1}^{(2)} + [\mathcal{C}_\mu(\lambda)]_{S_1ML_1}^{(2)} + [\mathcal{C}_\mu(\lambda)]_{S_1S_2M}^{(2)} \\
 &\quad + [\mathcal{C}_\mu(\lambda)]_{S_2'S_1M}^{(2)} + [\mathcal{C}_\mu(\lambda)]_{S_1M}^{(2)} + [\mathcal{C}_\mu(\lambda)]_{ML_1L_2}^{(2)} \\
 &\quad + [\mathcal{C}_\mu(\lambda)]_{ML_1}^{(2)} \\
 &= \mu^2 e^{-\lambda} \left[\frac{\pi^2}{3} - \ln 2(4 + \ln 2 + \gamma_E) \right. \\
 &\quad \left. + \lambda \left(-\frac{\pi^2}{6} + \ln 2(\ln 2 + \gamma_E) \right) \right]. \tag{164}
 \end{aligned}$$

V. HIERARCHICAL STRUCTURE OF THE IMPORTANT TRAPS

It is now clear that the procedure we have described up to order μ^2 can be generalized at an arbitrary order n : all observables at order μ^n can be obtained by considering a dispersion of the thermal packet over at most $(1+n)$ traps that have to be chosen among a certain number Ω_n of possible configurations of the traps. Our aim in this section is not to pursue any further explicit computations, but to get some insight into the set of important traps that play a role at a given order n .

A. Set of the important traps at order n

At order n , the important traps are the main trap M ; the following n large renormalized traps L_1, \dots, L_n ; the n biggest traps S_1, \dots, S_n among the small traps before M ; the $(n-1)$ biggest traps $I_2^{(1)}, \dots, I_n^{(1)}$ among the small traps in the interval between M and L_1 , the $(n-2)$ biggest traps $I_3^{(2)}, \dots, I_n^{(2)}$ among the small traps in the interval between L_1 and L_2 ; and so on; the biggest trap $I_n^{(n-1)}$ among the small traps in the interval between L_{n-2} and L_{n-1} .

The index at the bottom represents the order of occupation in μ as in the Fig. 1. The total number of traps is thus

$$T_n = 1 + n + \sum_{i=1}^n i = 1 + \frac{n(n+3)}{2}, \tag{165}$$

which generalizes $T_1=3$ (M, S_1, L_1) and $T_2=6$ (M, S_1, S_2 identified with S_2', L_1, L_2).

B. Set of the important configurations at order n

With these T_n traps, we have now to construct the possible Ω_n configurations of $(1+n)$ traps that are ordered by in positions, and that contribute up to order μ^n . We have

$$\Omega_n = \Omega_{n-1} + \omega_n = \sum_{i=0}^n \omega_i, \tag{166}$$

where ω_n represents the number of configurations that begin to contribute at order n .

We may now decompose

$$\omega_n = a_n^{(L_n)} + a_n^{(L_{n-1})} + \dots + a_n^{(L_1)} + a_n^M, \tag{167}$$

where $a_n^{(L_j)}$ is the number of configurations that contain L_j as the rightmost trap. For $j=n$, there is only $a_n^{(L_n)}=1$ configuration $ML_1L_2\cdots L_n$, whereas for $j=0$,

$$a_n^M = n!, \quad (168)$$

since we have to order in space the n traps S_1, \dots, S_n before M . More generally, at order j , to construct the configurations of $(n+1)$ traps containing $ML_1\cdots L_j$, which represent $(j+1)$ fixed traps, we have to choose $(n-j)$ traps among the $(j+1)$ available intervals and to count the possible positional orders in each interval as

$$a_n^{L_j} = \sum_{p_1=0}^{+\infty} \cdots \sum_{p_{j+1}=0}^{+\infty} \delta\left(\sum_{i=1}^{j+1} p_i = n-j\right) p_1! \cdots p_{j+1}! \quad (169)$$

The final result is thus that the number of new configurations that appear at order n reads

$$\omega_n = \sum_{j=0}^n \left[\sum_{p_1=0}^{+\infty} \cdots \sum_{p_{j+1}=0}^{+\infty} \delta\left(\sum_{i=1}^{j+1} p_i = n-j\right) p_1! \cdots p_{j+1}! \right], \quad (170)$$

which generalizes what we have found before for the lowest orders $\omega_0=1$ (M), $\omega_1=2$ (S_1M and ML_1), and $\omega_2=5$ (S_1S_2M , S_2S_1M , MI_1L_1 , ML_1L_2 , and S_1ML_1).

VI. QUANTITATIVE MAPPING BETWEEN THE BIASED SINAI DIFFUSION AND THE DIRECTED TRAP MODEL

A. Renormalized landscape for the biased Brownian motion

The real space renormalization group (RSRG) method can also be applied to the biased Brownian landscape [2,13]. The distributions of the barriers $F=\Gamma+\xi$ in the renormalized landscape at scale Γ are given by [2,13]

$$P_{\Gamma}^{+}(\xi) = \frac{2\delta}{e^{2\delta\Gamma}-1} \exp\left(-\xi \frac{2\delta}{e^{2\delta\Gamma}-1}\right) \simeq \frac{2\delta}{e^{2\delta\Gamma}} \exp\left(-\xi \frac{2\delta}{e^{2\delta\Gamma}}\right), \quad (171)$$

$$P_{\Gamma}^{-}(\xi) = \frac{2\delta}{1-e^{-2\delta\Gamma}} \exp\left(-\xi \frac{2\delta}{1-e^{-2\delta\Gamma}}\right) \simeq 2\delta e^{-\xi 2\delta}, \quad (172)$$

where the parameter 2δ reads in terms of the notations (1,5)

$$2\delta \equiv \frac{\mu}{T} = \frac{F_0}{\sigma}. \quad (173)$$

As a consequence, the distribution $P_{\Gamma}^{-}(\xi)$ of barriers against the bias can be considered as infinitely large only in the limit of vanishing bias $\delta \rightarrow 0$. It is only in this limit that all particles of the same thermal packet remain in the same renormalized valley asymptotically.

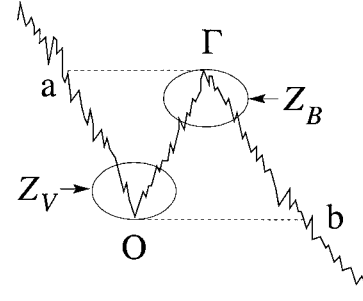


FIG. 2. Computation of the escape time from a renormalized valley of barrier Γ : we consider the first-passage time at b for a particle starting at 0 . The double integral (176) is dominated by the Arrhenius factor $e^{\beta\Gamma}$, and the prefactor is the product of two partition functions: Z_V represents the partition function of the bottom of the valley and Z_B represents the partition function of the inverse potential $-V$ near the top of the barrier Γ .

B. Trapping time of a renormalized valley of barrier Γ

Let us now recall a standard result for one-dimensional Fokker-Planck equation [21]: for a particle diffusing in a potential $U(x)$ on an interval $[a, b]$ with reflecting condition at a , the exit time defined as the first-passage time $\theta(x)$ at the point b for a particle starting at $x \in (a, b)$ at time $t=0$ can be studied for an arbitrary potential $U(x)$: the moments

$$\theta_n(x) \equiv \langle [\theta(x)]^n \rangle \quad (174)$$

are given by the recurrence

$$\theta_n(x) = \beta \int_x^b dy e^{\beta U(y)} \int_a^y dz e^{-\beta U(z)} [n \theta_{n-1}(z)], \quad (175)$$

with the initial condition $\theta_{n=0}(x) = 1$. In particular, the first moment reads

$$\theta_1(x) = \beta \int_x^b dy e^{\beta U(y)} \int_a^y dz e^{-\beta U(z)}. \quad (176)$$

For the biased Brownian landscape, the exit time over a barrier Γ when starting at the bottom of a renormalized valley that we choose as the origin can be obtained (see Fig. 2) by choosing a at the height Γ on the renormalized descending bond on the left and b at a potential Γ after the top of the barrier Γ . It seems that usually [21] one chooses b exactly at the top of the barrier to derive the Arrhenius factor, but we think that to obtain the correct prefactor, one has to choose b on the descending potential *after* the top to be sure that the particle will not return in the trap where it started. Indeed, when the particle sits just on the top, there is a finite probability to return to its starting trap, which is, for instance, a probability $1/2$ for a potential that is symmetric around its top. So for a given realization V of a renormalized valley, the first moment of the escape time reads

$$\theta_1\{V\} = \beta \int_0^b dy e^{\beta V(y)} \left[\int_a^0 dz e^{-\beta V(z)} + \int_0^y dz e^{-\beta V(z)} \right], \quad (177)$$

where the biased Brownian potential $V(x) = -F_0x + U(x)$ satisfies the constraints of a renormalized valley at scale Γ (see Fig. 2): it starts at $V(0) = \epsilon$, it then evolves on each side $x > 0$ and $x < 0$ in the presence of absorbing boundaries at 0 and Γ , and is conditioned to finish at $V = \Gamma$ and not at $V = 0$. On the negative side, $x = a$ is the random position where the potential first hit Γ . On the positive side, after the random position l_Γ where the potential first hit Γ , the potential again evolves in the presence of absorbing boundaries at 0 and Γ , and is conditioned to finish at $V = 0$ at some random position called b , and not at $V = \Gamma$.

As usual for the problem of escape over a large barrier, the double integral (177) is dominated by the saddle $V(y)$ is maximal and where $V(z)$ is minimal. For a renormalized valley (see Fig. 2), these regions are $y \sim l_\Gamma^{(1)}$, where $V(y) \sim \Gamma$ and $z \sim 0$, where $V(z) \sim 0$. This saddle-point analysis yields the following leading behavior:

$$\theta_1\{V\} \simeq_{\Gamma \rightarrow \infty} \tau_0(V) e^{\beta\Gamma}. \quad (178)$$

The prefactor is simply given by the product

$$\tau_0(V) = \beta Z_B Z_V, \quad (179)$$

where Z_V is the partition function of the infinitely deep renormalized valley,

$$Z_V = \lim_{\Gamma \rightarrow \infty} \left(\int_0^{l_\Gamma^{(2)}} dz e^{-\beta V_-(z)} + \int_0^{l_\Gamma^{(1)}} dz e^{-\beta V_+(z)} \right), \quad (180)$$

where the random potentials

$$V_+(x) = -F_0x + U_1(x), \quad (181)$$

$$V_-(x) = F_0x + U_2(x) \quad (182)$$

are defined in terms of two independent Brownian trajectories $U_1(x)$ and $U_2(x)$ (3) starting at $V_+(0) = \epsilon = V_-(0)$. The potentials $V_\pm(x)$ evolves in the presence of absorbing boundaries at 0 and Γ , and are conditioned to finish at $V = \Gamma$ and not at $V = 0$. $l_\Gamma^{(1)}$ and $l_\Gamma^{(2)}$ are the random times where V_\pm , respectively, first hit $V = \Gamma$.

Similarly, the factor Z_B is the partition function of an independent infinitely deep renormalized valley, which represents what happens in the vicinity of the top of the barrier Γ when considered with the change $V \rightarrow -V$ to transform it in a valley (see Fig. 2). For the biased Brownian landscape considered here, by symmetry, Z_B is simply an independent realization of the variable Z_V .

The same saddle-point analysis may be applied to higher moments given by the recurrence (175) to obtain

$$\theta_n\{V\} \simeq_{\Gamma \rightarrow \infty} n! (\tau_0(V) e^{\beta\Gamma})^n. \quad (183)$$

So for a given renormalized valley of barrier Γ , the escape time t is distributed exponentially as in the trap model as

$$f_{\theta_1\{V\}}(t) = \frac{1}{\theta_1\{V\}} e^{-t/\theta_1\{V\}}, \quad (184)$$

where the trapping time $\theta_1\{V\}$ (178) depends mostly on the barrier Γ via the usual Arrhenius factor $e^{\beta\Gamma}$, but also on the details of the structure of the valley near the bottom and near the top via the prefactor (179).

C. Distribution of the trapping time of renormalized valleys

We are now interested in the distribution of the trapping time $\theta_1\{V\}$ over the ensemble of renormalized valleys existing in the renormalized landscape at scale Γ . The distribution of the barriers is given by Eq. (172). So we have to study the statistics of the prefactor (179).

It is more convenient to work with dimensionless quantities by rewriting the partition functions as $Z_V = z_1 / (\sigma\beta^2)$, $Z_B = z_2 / (\sigma\beta^2)$ so that

$$\tau_0(V) = \frac{e^{\beta\Gamma}}{\sigma^2\beta^3} z_1 z_2, \quad (185)$$

where z_1 and z_2 are independent random variables whose probability distribution $P(z)$ is characterized in Appendix B by its Laplace transform

$$\int_0^{+\infty} dz e^{-sz} P(z) = \left[\frac{1}{\Gamma(1+\mu)} \frac{(\sqrt{s})^\mu}{I_\mu(2\sqrt{s})} \right]^2, \quad (186)$$

where I_μ is the Bessel function of index μ . In the renormalized landscape at scale Γ , the probability distribution of the trapping time τ of the renormalized valleys thus reads, using Eq. (172),

$$\begin{aligned} P_\Gamma(\tau) &= \int_0^{+\infty} d\xi 2\delta e^{-2\delta\xi} \int_0^{+\infty} dz_1 P(z_1) \int_0^{+\infty} dz_2 P(z_2) \\ &\times \delta \left[\tau - \frac{e^{\beta(\Gamma+\xi)}}{\sigma^2\beta^3} z_1 z_2 \right] \\ &= \frac{\mu}{\tau} \left(\frac{e^{\beta\Gamma}}{\sigma^2\beta^3 \tau} \right)^\mu \int_0^{+\infty} dz_1 z_1^\mu P(z_1) \int_0^{+\infty} dz_2 (z_2)^\mu P(z_2). \end{aligned} \quad (187)$$

So we have to compute the noninteger moment of order μ of the variable z . Using the integral representation valid for $0 < \mu < 1$,

$$z^\mu = \frac{\mu}{\Gamma(1-\mu)} \int_0^{+\infty} ds s^{-1-\mu} (1 - e^{-sz}), \quad (189)$$

we obtain the moment from the Laplace transform (186)

$$\begin{aligned} \int_0^{+\infty} dz z^\mu P(z) &= \frac{\mu}{\Gamma(1-\mu)} \int_0^{+\infty} ds s^{-1-\mu} \\ &\times \left(1 - \left[\frac{1}{\Gamma(1+\mu)} \frac{(\sqrt{s})^\mu}{I_\mu(2\sqrt{s})} \right]^2 \right) \end{aligned} \quad (190)$$

$$= \frac{2\mu 4^\mu}{\Gamma(1-\mu)} \int_0^{+\infty} dw \times \left[w^{-1-2\mu} - \frac{1}{\Gamma^2(1+\mu) 4^\mu w I_\mu^2(w)} \right]. \quad (191)$$

Using the Wronskian property of Bessel functions, and their series expansion at small argument, we finally get

$$\int_0^{+\infty} dz z^\mu P(z) = \frac{1}{\Gamma(1-\mu)} \lim_{a \rightarrow 0} \left[\left(\frac{a}{2} \right)^{-2\mu} - \frac{2\mu}{\Gamma^2(1+\mu)} \frac{K_\mu(a)}{I_\mu(a)} \right] = \frac{1}{\Gamma(1+\mu)}. \quad (192)$$

The final result is thus that the distribution of trapping time τ of the renormalized valleys existing at scale Γ reads, Eq. (188),

$$P_\Gamma(\tau) = \frac{\mu}{\tau} \left(\frac{e^{\beta\Gamma}}{\sigma^2 \beta^3 \tau} \right)^\mu \frac{1}{\Gamma^2(1+\mu)}. \quad (193)$$

D. Precise choice of the renormalization scale Γ as a function of time

We have seen in the trap model that the distribution of renormalized traps at t reads, Eq. (41),

$$q_t(\tau) = \theta(t < \tau) \frac{\mu}{\tau} \left(\frac{t}{\tau} \right)^\mu. \quad (194)$$

To make it exactly coincide with the distribution (193) of the biased Sinai model, we have to choose the renormalized scale Γ of the landscape to be the following function of time:

$$\Gamma(t) = T \ln \{ t \sigma^2 \beta^3 [\Gamma^2(1+\mu)]^{1/\mu} \}. \quad (195)$$

The RSRG method [2] gives that the distribution of the length l_+ of the descending renormalized bonds is simply exponential in the limit $\Gamma \rightarrow \infty$:

$$P_\Gamma(l_+) = \frac{1}{b_\Gamma} e^{-l_+/b_\Gamma}, \quad (196)$$

where the mean length reads

$$b_\Gamma = \frac{1}{\sigma(2\delta)^2} e^{2\delta\Gamma} = \frac{1}{\sigma\beta^2\mu^2} e^{2\delta\Gamma}, \quad (197)$$

so that it reads as a function of time (195) as

$$b(t) = b_{\Gamma(t)} = \frac{\Gamma^2(1+\mu)}{\sigma\beta^2\mu^2} [t\sigma^2\beta^3]^\mu = \frac{\Gamma^2(\mu)}{\sigma\beta^2} [t\sigma^2\beta^3]^\mu. \quad (198)$$

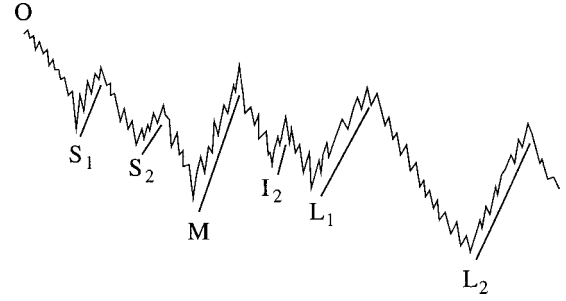


FIG. 3. Hierarchical structure of the important valleys for a particle starting at the origin. The barriers against the bias, which are emphasized by the straight lines correspond to the depths of the trap model represented in Fig. 1. The bottom M of the renormalized valley that contains the origin at scale Γ is occupied with a weight $O(\mu^0)$. The bottom L_1 of the next renormalized valley and the bottom S_1 of the biggest subvalley before M are occupied with weights $O(\mu)$. The next-nearest renormalized valley L_2 , the biggest subvalley I_2 between M and L_1 , and the second biggest subvalley S_2 before M are occupied with weights of order $O(\mu^2)$.

This length scale $b(t)$ exactly corresponds to the ratio $t^\mu c_{trap}(\mu)/c_{sinai}(\mu)$ of the constants appearing in the exact diffusion front of the two models (A24).

E. Usual RSRG in the limit $\mu \rightarrow 0$

It has been shown in Ref. [2] that the “effective dynamics,” where at time t , the particle is typically at time t around the minimum of the renormalized valley containing the initial condition, is sufficient in the double limit $t \rightarrow \infty$ $\delta \rightarrow 0$ with

$$\gamma \equiv \delta\Gamma(t) \quad (199)$$

fixed and $X = x/\Gamma^2(t)$ fixed.

The limit $\gamma \rightarrow 0$ corresponds to the symmetric Sinai diffusion, whereas in the limit $\gamma \rightarrow \infty$, the model becomes directed at large scale and the diffusion front converges towards [2]

$$\overline{P(x,t|0,0)}_{\gamma \rightarrow \infty} \approx \theta(x) \frac{1}{b(t)} e^{-x/b(t)}, \quad (200)$$

where $b(t)$ represents the mean length of renormalized descending bonds (198). This limit actually corresponds to the limit $\mu \rightarrow 0$ of the exact Lévy front [14,17,18] as described in Appendix.

F. Spreading of the thermal packet over many renormalized valleys

The renormalized valleys of the Sinai model with bias are the analog of the traps in the directed model (see Fig. 3). For $\mu \rightarrow 0$, the bottom of the renormalized valley containing the origin described above, Eq. (200) is the analog of the main trap M described in Sec. II.

At first order in μ , as in Sec. III A, there are the following two effects.

Next renormalized valley L_1 . The main renormalized valley M at scale $\Gamma(t)$ has a trapping time τ_M that is distributed

as in the directed trap model (41), since we have defined the relation $\Gamma(t)$ (195) by identifying the trapping-time distributions of the two models. So there is a small probability $(1 - e^{-t/\tau_M})$ that the particle has already escaped from this main renormalized M at time t to jump into the next renormalized valley L_1 .

Moreover, the RSRG approach [2] yields that the joint distribution of the trapping time τ_M and of the positions x_M and x_L of the bottoms of the main renormalized valley and of the the next renormalized valley L_1 reads

$$\mathcal{D}_{M,L_1}(x,x_L;\tau_M) = \theta(t < \tau_M) \theta(0 < x < x_L) \times \frac{\mu t^\mu}{\tau_M^{1+\mu}} \frac{1}{b^2(t)} e^{-x/b(t)} e^{-(x_L-x)/b(t)} \quad (201)$$

$$= \theta(t < \tau_M) \theta(0 < x < x_L) \times \frac{\mu t^\mu}{\tau_M^{1+\mu}} \frac{1}{b^2(t)} e^{-x_L/b(t)}, \quad (202)$$

which is the analog of Eq. (39). The only change is in the prefactor in front of t^μ in the scale $b(t)$ (198).

Last decimated renormalized valley S_1 . The last decimated barrier against the bias inside the main renormalized valley between the origin and the bottom defines the last decimated subvalley S_1 : it has a trapping time $\tau_{S_1} < t$ that is not zero and thus there is a small probability $e^{-t/\tau_{S_1}}$ that the particle is still trapped in the subvalley S_1 at time t .

Moreover, the RSRG approach [2] yields that the joint distribution of the trapping time τ_S and of the positions x_S and x of the bottoms of the last decimated valley S_1 and of the the main renormalized valley M reads

$$\mathcal{D}_{S_1,M}(x_S,x;\tau_S) = \theta(t > \tau_S) \theta(0 < x_S < x) \times \frac{\mu}{\tau_S} \frac{1}{b(t)b(\tau_S)} e^{-x/b(\tau_S)}, \quad (203)$$

which is the analog of Eq. (46). The only change is again in the prefactor in the scale $b(t)$ (198).

It is clear that this analysis may be generalized to further orders in μ .

G. Conclusion: Equivalence of the two large-scale renormalized descriptions

The statistical properties of the spreading of the thermal packet over many renormalized valleys and subvalleys inside the main one are thus exactly the same as in the directed trap model discussed in details in previous sections. In particular, the localization parameters Y_k of the trap model represent coarse-grained localization parameters for the biased Sinai diffusion: ‘‘at the same position’’ in the trap model means ‘‘at a finite distance around the bottom of the same renormalized valley’’ for the biased Sinai diffusion. As a consequence, for all rescaled quantities x/t^μ , the results are exactly the same up to the global prefactor in the scale $b(t)$ (198): this

was already known for the averaged diffusion front (see Appendix A), but this also holds for the thermal width (28), for all other rescaled thermal cumulants (25), and for the long-range part of the two-point correlation function (30).

The difference of the Sinai model with respect to the directed trap model is thus the internal structure of a renormalized valley that induces a dispersion over finite distances of the particles that are in the same renormalized valley. We now study the statistical properties of the biased Brownian valleys.

VII. INTERNAL STRUCTURE OF THE TRAPS IN THE BIASED SINAI DIFFUSION

A. Probability distribution inside a renormalized valley

The probability distribution of particles inside the same renormalized valley can be obtained by generalizing the approach of the Sinai symmetric case [5]: for each realization of a renormalized valley, it is given by the Boltzmann distribution on this valley. So asymptotically as $t \rightarrow \infty$, the probability distribution of the distance y to the bottom of the valley averaged over the environment reads

$$P_V(y > 0) = \lim_{\Gamma \rightarrow \infty} \left\langle \frac{e^{-\beta V_+(y)}}{\int_0^{\Gamma^{(1)}} dx e^{-\beta V_+(x)} + \int_0^{\Gamma^{(2)}} dx e^{-\beta V_-(x)}} \right\rangle_{\{V_+\}, \{V_-\}},$$

$$P_V(y < 0) = \lim_{\Gamma \rightarrow \infty} \left\langle \frac{e^{-\beta V_-(|y|)}}{\int_0^{\Gamma^{(1)}} dx e^{-\beta V_+(x)} + \int_0^{\Gamma^{(2)}} dx e^{-\beta V_-(x)}} \right\rangle_{\{V_+\}, \{V_-\}}, \quad (204)$$

where the random potentials V_\pm satisfy the same conditions as in Eq. (182).

The computation of the functionals (204) is given in Appendix B. It yields the nonintuitive result that the probability distribution $P_V(y)$ is actually symmetric in $y \rightarrow -y$. The restoration of this symmetry comes from the conditioning of the biased random walk to reach Γ on each side. Its Laplace transform reads, Eq. (A17),

$$\hat{P}_V(p) \equiv \int_0^{+\infty} du e^{-pu} \hat{P}_V(y)$$

$$= \frac{1}{\Gamma^2(1+\mu)} \int_0^\infty ds \frac{\left(\frac{s}{2}\right)^{2\mu-1}}{I_\mu(s) I_\nu(s)} \int_0^s dz z I_\nu(z) \times \left[K_\mu(z) - \frac{K_\mu(s)}{I_\mu(s)} I_\mu(z) \right], \quad (205)$$

where the only factor containing the Laplace parameter is the index

$$\nu \equiv \sqrt{\mu^2 + \frac{4T^2 p}{\sigma}} = \mu + \frac{2T^2 p}{\sigma \mu} + O(p^2). \quad (206)$$

For $\mu > 0$, the series expansion in the Laplace parameter p is thus regular leading to

$$\hat{P}_V(p=0) = \frac{1}{2} - \frac{2T^2 p}{\mu \sigma} D(\mu) + \dots \quad (207)$$

All moments are thus finite, contrary to the symmetric case $\mu = 0$ [5], where the behavior as $(1 - \sqrt{pc} + \dots)$ corre-

sponds to the algebraic decay as $1/y^{3/2}$. So in the biased case, the distribution inside a renormalized valley is very narrow, contrary to the symmetric case.

B. Localization parameters inside a renormalized valley

For k particles that are in the same renormalized valley, the localization parameters may be computed as an average of the k th power of the local Boltzmann weight over the infinitely deep biased Brownian valleys (182). Generalizing the approach of Ref. [5] to the biased case, we have

$$(Y_k)_{valley} \sum_{\epsilon=\pm} \int_0^{+\infty} dy \left\langle \left(\frac{e^{-\beta V_\epsilon(|y|)}}{\int_0^{+\infty} dx e^{-\beta V_+(x)} + \int_0^{+\infty} dx e^{-\beta V_-(x)}} \right)^k \right\rangle_{\{V_+, V_-\}} \quad (208)$$

$$= \sum_{\epsilon=\pm} \frac{1}{\Gamma(k)} \int_0^{+\infty} dq q^{k-1} \langle e^{-q \int_0^{+\infty} dx e^{-\beta V_-(x)}} \rangle \left\langle \int_0^{+\infty} dy e^{-k\beta V_\epsilon(y)} e^{-q \int_0^{+\infty} dx e^{-\beta V_\epsilon(x)}} \right\rangle. \quad (209)$$

Using the results (A17) of the Appendix, we finally get

$$(Y_k)_{valley} = \frac{2}{\Gamma(k)\Gamma^2(1+\mu)} \left(\frac{\sigma\beta^2}{4} \right)^{k-1} \int_0^{+\infty} ds \frac{\left(\frac{s}{2}\right)^{2\mu-1}}{I_\mu^2(s)} \int_0^s dz z^{2k-1} I_\mu(z) \left[K_\mu(z) - \frac{K_\mu(s)}{I_\mu(s)} I_\mu(z) \right]. \quad (210)$$

C. Correlation function inside a renormalized valley

The correlation function of two particles at Boltzmann equilibrium in an infinitely deep biased Brownian valley reads

$$C_{valley}(l>0) = 2 \sum_{\epsilon=\pm} \int_0^\infty dy \left\langle \frac{e^{-\beta V_\epsilon(y) - \beta V_\epsilon(y+l)}}{\left(\int_0^\infty dx e^{-\beta V_+(x)} + \int_0^\infty dx e^{-\beta V_-(x)} \right)^2} \right\rangle + 2 \int_0^l dy \left\langle \frac{e^{-\beta V_+(y) - \beta V_-(l-y)}}{\left(\int_0^{+\infty} dx e^{-\beta V_+(x)} + \int_0^{+\infty} dx e^{-\beta V_-(x)} \right)^2} \right\rangle, \quad (211)$$

where the average $\langle \dots \rangle$ is over the realizations (V_+, V_-) satisfying Eq. (182).

Using the explicit results of Appendix A, we finally get

$$\begin{aligned} \hat{C}_{valley}(p) &= \frac{8}{\Gamma^2(1+\mu)} \int_0^\infty \frac{ds}{s} \frac{\left(\frac{s}{2}\right)^{2\mu}}{I_\mu^2(s)} \int_0^s dz_1 z_1 I_\mu(z_1) \int_0^s dz_2 z_2 \left(K_\mu(z_2) - \frac{K_\mu(s)}{I_\mu(s)} I_\mu(z_2) \right) \left[\theta(z_2 - z_1) I_\nu(z_1) \right. \\ &\quad \times \left(K_\nu(z_2) - \frac{K_\nu(s)}{I_\nu(s)} I_\nu(z_2) \right) + \theta(z_1 - z_2) I_\nu(z_2) \left(K_\nu(z_1) - \frac{K_\nu(s)}{I_\nu(s)} I_\nu(z_1) \right) \left. \right] + \frac{4}{\Gamma^2(1+\mu)} \\ &\quad \times \int_0^\infty \frac{ds}{s} \frac{\left(\frac{s}{2}\right)^{2\mu}}{I_\nu^2(s)} \left[\int_0^s dz z I_\nu(z) \left(K_\mu(z) - \frac{K_\mu(s)}{I_\mu(s)} I_\mu(z) \right) \right]^2. \end{aligned} \quad (212)$$

VIII. DISCUSSION OF THE UNIVERSALITY

We now briefly discuss the question of the universality. The RSRG method that describes the large-scale structure at scale Γ of the random potential is valid for all discrete models with random forces [2]. The parameter 2δ that describes the distribution of barriers against the drift at large scale (172) may be expressed for a discrete random force model as the nonzero solution of the equation [2]

$$\overline{e^{-2\delta f}} = 1, \quad (213)$$

which is known to determine the anomalous diffusion exponent $\mu = 2\delta T$ [15,16]. So for a given value of the parameters $(2\delta, \sigma)$, the renormalized landscape at scale Γ is universal.

However, it is clear from the analysis of the escape time of a renormalized valley (178) that the prefactor (179) in front of the Arrhenius factor $e^{\beta\Gamma}$ is not universal: the partition functions Z_V and Z_B depend on the details over finite scales of the potential near a bottom of a renormalized valley and near a top of a barrier.

So for a potential that belongs to the universality class $(2\delta, \sigma)$, but that is not a biased Brownian at small scales, the distribution of the trapping times in the renormalized landscape at scale Γ reads

$$P_\Gamma(\tau) = \frac{\mu}{\tau} \left(\frac{\beta e^{\beta\Gamma}}{\tau} \right)^\mu (\bar{Z}_V^\mu)(\bar{Z}_B^\mu), \quad (214)$$

so that the quantitative mapping onto the trap model (194) is realized for the choice of the RG scale Γ as a function of t according to

$$\Gamma(t) = T \ln \left[\frac{t}{\beta (\bar{Z}_V^\mu)^{1/\mu} (\bar{Z}_B^\mu)^{1/\mu}} \right], \quad (215)$$

which corresponds to the length scale

$$b(t) = b_{\Gamma(t)} = \frac{1}{\sigma \beta^2 \mu^2} \frac{t^\mu}{\beta^\mu (\bar{Z}_V^\mu)(\bar{Z}_B^\mu)}. \quad (216)$$

This shows that the factor μ^2 is universal and comes from the mean length of descending bonds in the renormalized landscape at large scale, whereas the factor $\Gamma^2(1+\mu)$ of the biased Brownian motion is not universal and comes from the probability distribution of the partition function of a biased Brownian valley (192). However, it is expected to be valid for discrete models in the limit where the lattice constant is very small as compared to the thermal length $l_T = T^2/\sigma$. For the localization parameters and the correlation function of two particles inside the same renormalized valley, the discussion of the universality is the same as in the symmetric case [5].

IX. CONCLUSIONS AND PERSPECTIVES

To study the anomalous diffusion phase $x \sim t^\mu$ of the directed trap model and of the Sinai diffusion with bias, we have extended the usual RSRG method that assumes a full localization in a single valley to allow for the spreading of

the thermal packet over many renormalized valleys. We have shown how all observables can be computed via a series expansion in μ : at any given order μ^n , it is sufficient to consider the spreading over at most $(1+n)$ traps. We have given explicit rules for the statistical properties of these traps. We have shown the exactness of these expansions in μ by comparing up to order $n=2$ with the already known exact results, such as the diffusion front [14], the thermal width [19], and the localization parameter Y_2 [20]. Our construction moreover gives a clear physical picture of the localization properties in the anomalous diffusion phase, and explains the typical shape of the diffusion front in a given sample obtained by numerical simulation (Fig. 4 of Ref. [20]).

In a forthcoming paper [22], we will adapt our method to study the localization properties and the aging behavior in the symmetric (i.e., undirected) trap model which has attracted a lot of interest recently [23–25].

For the field of biased diffusion in one-dimensional random potentials, it would be very interesting to study the influence of correlations on the localization properties studied here for the Brownian case. In particular, the case of algebraic correlations $(U(x) - U(y))^2 \sim |x - y|^\gamma$ is known to give rise to a creep motion for $0 < \gamma < 1$ [26]. For DNA sequences, it seems that the interesting cases are not only the Brownian case $\gamma = 1$ [9] but also the values $\gamma > 1$ [27]. Another physically interesting case concerns the logarithmic correlations, which give rise to a freezing transition in the dynamics [28] as well as in the statics [29].

From the point of view of the RSRG method, since the usual RSRG is asymptotically exact for infinite-disorder fixed points [13], the extension introduced here can be seen as a systematic expansion in the inverse disorder strength. It can therefore be used for the classical random field Ising chain in the presence of a small magnetic external field [7], as well as in the field of random quantum spin chains [13] to study the Griffiths phases. In particular, the RSRG decimation rules are the same for the random walks with random forces [2] and for the random transverse-field Ising spin chain (RTFIC) [13] presenting random fields h_i and random couplings J_i with the following dictionary: the forces f_i^- of the descending bonds correspond to $\ln(1/h_i)$, whereas the forces f_i^+ of the ascending bonds correspond to $\ln(1/J_i)$. The exponent 2δ defined in Eq. (213), which is conserved by the RG flow [Eqs. (28) and (31) in Ref. [2]] exactly corresponds for the random transverse-field Ising spin chain to the RG-invariant exponent 2Δ defined by [30]

$$1 = \left(\frac{J}{h} \right)^{2\Delta} = \overline{e^{2\Delta(\ln J - \ln h)}}. \quad (217)$$

Of course, the interesting observables in the two models are very different, so we will analyze elsewhere [31] the physics of the RTFIC for finite Δ beyond the regime $\Delta \rightarrow 0$, where the RSRG is asymptotically exact [13].

Finally, the expansion in the important traps for the dynamical models discussed in this paper has a static counterpart with the following differences: in the static case, the

expansion parameter is the temperature T , and the main trap M corresponds to the absolute minimum of the random potential. We have already shown in Ref. [32] for the toy model consisting of a Brownian potential plus a quadratic potential, how the thermal cumulants at first order in T can be explained by studying the statistical properties of the configurations presenting two nearly degenerate minima. We will discuss in Ref. [33] in a more general context the structure of the low-temperature series expansions in some disordered systems.

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APPENDIX A: USEFUL PROPERTIES OF THE LÉVY DIFFUSION FRONT FOR $0 < \mu < 1$

In this appendix, we recall some useful properties of the Lévy distributions ([14,34], and references therein).

1. Definition and properties of one-sided Lévy stable laws

The rescaled sum

$$y = \frac{1}{n^{1/\mu}} \sum_{i=1}^n t_i \quad (\text{A1})$$

of n identical independent positive random variables distributed with a law presenting the algebraic decay

$$p(t) \underset{t \rightarrow \infty}{\simeq} \frac{A}{t^{1+\mu}}, \quad (\text{A2})$$

where $0 < \mu < 1$, has for limit distribution as $n \rightarrow \infty$ the one-sided Lévy law $L_{\mu,c(\mu;A)}(y)$ defined by its Laplace transform

$$\int_0^{+\infty} dy e^{-sy} L_{\mu,c}(y) = e^{-cs^\mu}, \quad (\text{A3})$$

and where the constant c reads

$$c(\mu;A) = \frac{\pi A}{\sin \pi \mu \Gamma(1+\mu)}. \quad (\text{A4})$$

In this paper, we will only use the following series representation [14,34]:

$$L_{\mu,c}(y) = -\frac{1}{\pi y} \sum_{k=1}^{+\infty} \left(-\frac{c}{y^\mu} \right)^k \frac{\Gamma(1+k\mu)}{\Gamma(1+k)} \sin \pi \mu k, \quad (\text{A5})$$

which is convergent in the whole phase $0 < \mu < 1$.

We stress here that we have defined the constant c by the Laplace transform (A3). Writing the inverse Laplace transform as a Fourier integral yields

$$L_{\mu,c}(y) = \int_{-i\infty}^{+i\infty} \frac{ds}{2i\pi} e^{-sy - cs^\mu} = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} \times \exp \left[-ity - ct^\mu \left(\cos \frac{\pi\mu}{2} + i \operatorname{sgn} \sin \frac{\pi\mu}{2} \right) \right], \quad (\text{A6})$$

so that the constant \mathcal{C} appearing in the usual Fourier transform of Lévy distributions,

$$L_{\mu,c}(y) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} \exp \left[-ity - \mathcal{C}t^\mu \left(1 + i \operatorname{sgn}(t) \tan \frac{\pi\mu}{2} \right) \right] \quad (\text{A7})$$

reads in terms of the Laplace constant c ,

$$\mathcal{C} = c \cos \frac{\pi\mu}{2}. \quad (\text{A8})$$

2. Lévy diffusion front for the trap model

For a given trap τ , the distribution of the escape time t is exponential (9), which yields after averaging over τ (10),

$$\begin{aligned} \bar{f}_\tau(t) &= \int_0^{+\infty} d\tau q(\tau) f_\tau(t) \\ &= \int_0^{+\infty} \frac{dv}{v} q\left(\frac{t}{v}\right) e^{-v} \underset{t \rightarrow \infty}{\simeq} \frac{\mu \Gamma(1+\mu)}{t^{1+\mu}}. \end{aligned} \quad (\text{A9})$$

For a given sample (τ_0, τ_1, \dots) , the probability $P_t(n)$ for the particle to be in the trap n at time t reads

$$\begin{aligned} P_t(n) &= \int \prod_{i=0}^{+\infty} dt_i f_{\tau_i}(t_i) \\ &\times \theta(t_0 + t_1 + \dots + t_{n-1} < t < t_0 + t_1 + \dots + t_n). \end{aligned} \quad (\text{A10})$$

The average over the disorder

$$\begin{aligned} \overline{P_t(n)} &= \int \prod_{i=0}^{+\infty} dt_i \bar{f}_\tau(t_i) \\ &\times \theta(t_1 + t_2 + \dots + t_{n-1} < t < t_1 + t_2 + \dots + t_n) \end{aligned} \quad (\text{A11})$$

shows that the diffusion front at large time is directly related to the properties of the sum of a large number n of independent variables t_i distributed with the law (A9) presenting an algebraic decay (A2): the rescaled variable $y = t/n^{1/\mu}$ is distributed with a one-sided stable Lévy distribution $L_{\mu,c_{\text{trap}}(\mu)}$ (A3), where the constant $c_{\text{trap}}(\mu)$ reads for the case (A9) and (A4),

$$c_{\text{trap}}(\mu) = \frac{\pi \mu}{\sin \pi \mu}. \quad (\text{A12})$$

The variable $X = n/t^\mu = y^{-\mu}$ is thus distributed with the law

$$f_{\mu,c}(X) = \frac{1}{\mu X^{1+1/\mu}} L_{\mu,c}(X^{-1/\mu}) \quad (\text{A13})$$

with the special value $c = c_{\text{trap}}(\mu)$.

In particular, the series expansion (A5) gives the following series representation for the diffusion front:

$$\begin{aligned} f_{\mu,c}(X) &= \frac{c}{\pi\mu} \sum_{k=1}^{+\infty} (-cX)^{k-1} \frac{\Gamma(1+k\mu)}{\Gamma(1+k)} \sin \pi\mu k \\ &= c \sum_{k=1}^{+\infty} \frac{(-cX)^{k-1}}{(k-1)! \Gamma(1-k\mu)}. \end{aligned} \quad (\text{A14})$$

Using the series expansion

$$\frac{1}{\Gamma(1-z)} = 1 + \sum_{m=1}^{+\infty} d_m (-1)^m z^m \quad (\text{A15})$$

with the first coefficient

$$d_1 = \gamma_E, \quad (\text{A16})$$

$$d_2 = \frac{\gamma_E^2}{2} - \frac{\pi^2}{12}, \quad (\text{A17})$$

where γ_E denotes the Euler's constant, we get the expansion in μ of the series (A14) for fixed c ,

$$\begin{aligned} f_{\mu,c}(X) &= c e^{-cX} [(1 - d_1\mu + d_2\mu^2) + (d_1\mu - 3d_2\mu^2)cX \\ &\quad + d_2\mu^2(cX)^2 + O(\mu^3)]. \end{aligned} \quad (\text{A18})$$

Expanding also in μ the value (A12)

$$c_{\text{trap}}(\mu) = 1 + \frac{\pi^2}{6} \mu^2 + O(\mu^3), \quad (\text{A19})$$

we get that the diffusion front reads up to second order in μ ,

$$\begin{aligned} g(X) = f_{\mu,c_{\text{trap}}(\mu)}(X) &= \left[1 - d_1\mu + \left(d_2 + \frac{\pi^2}{6} \right) \mu^2 \right] e^{-X} \\ &\quad + \left[d_1\mu - \left(3d_2 + \frac{\pi^2}{6} \right) \mu^2 \right] X e^{-X} + d_2\mu^2 X^2 e^{-X} \\ &\quad + O(\mu^3). \end{aligned} \quad (\text{A20})$$

Using the numerical values (A18), we finally get the expression (15) of the text.

3. Lévy diffusion front for the biased Sinai model

The exact form of the diffusion front was first determined in Ref. [15] for a corresponding discrete model. For the continuum model, the result has been proved in Theorem 1 of Ref. [17], which states that for $0 < \mu < 1$, the rescaled variable $X = x/t^\mu$ for probability distribution (A13), where the constant $c_s(\mu)$ is given by a complicated implicit expression

in Ref. [17]. The value of this constant has been proved in Ref. [18] to have the following simple expression:

$$c_{\text{sinai}}(\mu) = 8^\mu \frac{\pi\mu}{2\Gamma^2(\mu) \sin \pi\mu}, \quad (\text{A21})$$

where we have used Eq. (A8).

In Ref. [14], the same form was conjectured from the heuristic equivalence with the directed trap model via an identification of the parameters on some observable

$$c_{\text{sinai}}(\mu) = \frac{2}{x_1} \left(\frac{\tau_1}{2} \right)^\mu \frac{\pi\mu}{\Gamma^2(\mu) \sin \pi\mu}, \quad (\text{A22})$$

where $x_1 = 2T^2/\sigma$ and $\tau_1 = x_1^2/(2D_0)$ in terms of the diffusion constant D_0 in the pure case. This expression indeed coincides with Eq. (A22) for the units $T=1$, $\sigma=\frac{1}{2}$, and $D_0=\frac{1}{2}$ used in Ref. [18].

In the notations used in this paper $D_0=T$ (4), this corresponds to

$$c_{\text{sinai}}(\mu) = \frac{\sigma\beta^2}{(\sigma^2\beta^3)^\mu} \frac{\pi\mu}{\Gamma^2(\mu) \sin \pi\mu}. \quad (\text{A23})$$

To compare with the directed trap model, it is convenient to consider the ratio of the two constants (A12),

$$\frac{c_{\text{sinai}}(\mu)}{c_{\text{trap}}(\mu)} = \frac{\sigma\beta^2}{(\sigma^2\beta^3)^\mu} \frac{1}{\Gamma^2(\mu)}. \quad (\text{A24})$$

So beyond the natural dimensional factors, there is still a function $\Gamma^2(\mu)$ between the two models, whose origin will be discussed in details in the text.

APPENDIX B: STATISTICS OF THE INTERNAL STRUCTURE OF RENORMALIZED VALLEYS

1. Distribution inside a renormalized valley

To compute the functionals (204), we generalize the approach developed in Ref. [5] for the symmetric case $\mu=0$. We first exponentiate the denominator

$$P_\infty(y>0) = \int_0^\infty dq R_\infty^-(q) S_\infty^+(y,q), \quad (\text{B1})$$

$$P_\infty(y<0) = \int_0^\infty dq R_\infty^+(q) S_\infty^-(y,q), \quad (\text{B2})$$

where

$$R_\Gamma^\pm(q) \equiv \left\langle \exp \left[-q \int_0^{l_\Gamma} dx e^{-\beta V_\pm(x)} \right] \right\rangle_{\{V_\pm\}}, \quad (\text{B3})$$

$$S_\Gamma^\pm(y,q) \equiv \left\langle e^{-\beta V_\pm(y)} \exp \left[-q \int_0^{l_\Gamma} dx e^{-\beta V_\pm(x)} \right] \right\rangle_{\{V_\pm\}}. \quad (\text{B4})$$

These two functionals may be expressed as

$$R_{\Gamma}^{\pm}(q) = \mathcal{N}_{\pm} \int_0^{+\infty} dl \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} F_{[0,\Gamma]}^{\pm}(\Gamma - \epsilon, l | \epsilon), \quad (\text{B5})$$

$$S_{\Gamma}^{\pm}(y, q) = \mathcal{N}_{\pm} \int_0^{\Gamma} du \int_0^{+\infty} dl \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} F_{[0,\Gamma]}^{\pm} \\ \times (\Gamma - \epsilon, l | u) e^{-\beta u} F_{[0,\Gamma]}^{\pm}(u, y | \epsilon) \quad (\text{B6})$$

in terms of the path integrals

$$F_{[0,\Gamma]}^{\pm}(u, l | u_0) = \int_{V(0)=u_0}^{V(l)=u} \mathcal{D}V(x) \exp \left[-\frac{1}{4\sigma} \int_0^l dx \left(\frac{dV}{dx} \pm F_0 \right)^2 \right. \\ \left. - q \int_0^l dx e^{-\beta V(x)} \right] \Theta_{[0,\Gamma]} \{V(x)\}, \quad (\text{B7})$$

where the symbol $\Theta_{[0,\Gamma]} \{V(x)\}$ means that there are absorbing boundaries at $V=0$ and $V=\Gamma$. The expansion of the quadratic term of the measure yields

$$F_{[0,\Gamma]}^{\pm}(u, l | u_0) = e^{-(F_0^2/4\sigma)l \mp \delta(u-u_0)} F_{[0,\Gamma]}^{\pm}(u, l | u_0), \quad (\text{B8})$$

where

$$F_{[0,\Gamma]}^{\pm}(u, l | u_0) = \int_{V(0)=u_0}^{V(l)=u} \mathcal{D}V(x) \exp \left[-\frac{1}{4\sigma} \int_0^l dx \left(\frac{dV}{dx} \right)^2 \right. \\ \left. - q \int_0^l dx e^{-\beta V(x)} \right] \Theta_{[0,\Gamma]} \{V(x)\} \quad (\text{B9})$$

represents the analogous path integral for the symmetric case. Its Laplace transform has been computed in Eq. (B18) of Ref. [5]. We get

$$R_{\Gamma}^{\pm}(q) = \frac{\mathcal{N}_{\pm} e^{\mp \delta \Gamma}}{\sigma E \left(0, \Gamma, \frac{F_0^2}{4\sigma} \right)} \quad (\text{B10})$$

and the Laplace transforms with respect to y ,

$$\hat{S}_{\Gamma}^{\pm}(p, q) \equiv \int_0^{+\infty} dy e^{-py} S_{\Gamma}^{\pm}(y, q) = \frac{\mathcal{N}_{\pm} e^{\mp \delta \Gamma}}{\sigma^2} \int_0^{\Gamma} du e^{-\beta u} \\ \times \frac{E \left(0, u, \frac{F_0^2}{4\sigma} \right) E \left(u, \Gamma, p + \frac{F_0^2}{4\sigma} \right)}{E \left(0, \Gamma, \frac{F_0^2}{4\sigma} \right) E \left(0, \Gamma, p + \frac{F_0^2}{4\sigma} \right)}, \quad (\text{B11})$$

in terms of the function

$$E(u, v, p) = \frac{2}{\beta} \left[I_{(2/\beta)\sqrt{p/\sigma}} \left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} e^{-\beta u/2} \right) \right. \\ \times K_{(2/\beta)\sqrt{p/\sigma}} \left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} e^{-\beta v/2} \right) \\ \left. - K_{(2/\beta)\sqrt{p/\sigma}} \left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} e^{-\beta u/2} \right) \right. \\ \left. \times I_{(2/\beta)\sqrt{p/\sigma}} \left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} e^{-\beta v/2} \right) \right]. \quad (\text{B12})$$

The normalizations \mathcal{N}_{\pm} are obtained with the conditions $R_{\Gamma}^{\pm}(q \rightarrow 0) = 1$,

$$\mathcal{N}_{\pm} = \sigma \frac{\sinh \delta \Gamma}{\delta} e^{\pm \delta \Gamma}. \quad (\text{B13})$$

We thus obtain that there is no dependence in the sign \pm for the functionals $R_{\Gamma}^{\pm}(q)$ and $\hat{S}_{\Gamma}^{\pm}(p, q)$. As a consequence, we get the nonintuitive result that the probability distribution $P_V(y)$ is symmetric in $y \rightarrow -y$. The restoration of this symmetry comes from the conditioning of the random walk to reach Γ . We note that similarly, the distribution of the random times I_{Γ}^+ and I_{Γ}^- are also the same, since we have, with the notations of [2]

$$\frac{P_{\Gamma}^{\pm}(\zeta=0, s)}{P_{\Gamma}^{\pm}(\zeta=0, 0)} = \frac{U_{\Gamma}^{\pm}(s)}{U_{\Gamma}^{\pm}(0)} = \frac{\sqrt{s + \delta^2} \sinh \delta \Gamma}{\delta \sinh \Gamma \sqrt{s + \delta^2}}. \quad (\text{B14})$$

The Laplace transform of the distribution inside a valley thus reads

$$\hat{P}_V(p) \equiv \int_0^{+\infty} dy e^{-py} P_V(y > 0) = \int_0^{\infty} dq R_{\infty}^{\pm}(q) S_{\infty}^{\pm}(p, q), \quad (\text{B15})$$

$$R_{\infty}^{\pm}(q) = \frac{1}{\Gamma(1+\mu)} \frac{\left(\frac{1}{\beta} \sqrt{\frac{q}{\sigma}} \right)^{\mu}}{I_{\mu} \left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} \right)}, \quad (\text{B16})$$

$$\hat{S}_{\infty}^{\pm}(p, q) = \frac{\left(\frac{1}{\beta} \sqrt{\frac{q}{\sigma}} \right)^{\mu}}{q \Gamma(1+\mu)} \int_0^{2/\beta} \sqrt{q/\sigma} dz z \\ \times \frac{I_{\sqrt{\mu^2 + 4T^2 p/\sigma}}(z)}{I_{\sqrt{\mu^2 + 4T^2 p/\sigma}} \left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} \right)} \\ \times \left[K_{\mu} \left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} \right) - \frac{K_{\mu}(z) - \frac{I_{\mu}(z)}{I_{\mu} \left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} \right)}}{I_{\mu} \left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}} \right)} \right]. \quad (\text{B17})$$

The final result is thus given by Eq. (205) of the text.

2. Partition function of a renormalized valley

We now consider the probability distribution of the partition function of a renormalized valley (180),

$$Z_V = \int_0^{+\infty} dz e^{-\beta V_-(z)} + \int_0^{+\infty} dz e^{-\beta V_+(z)}, \quad (\text{B18})$$

where the potentials satisfy the constraints (182).

Its Laplace transform can be directly expressed in terms of the functions (B3), which we have computed before Eq. (B17),

$$\int_0^{+\infty} dZ_V \mathcal{P}(Z_V) e^{-qZ_V} = R_\infty^+(q) R_\infty^-(q) = \left[\frac{1}{\Gamma(1+\mu)} \frac{\left(\frac{1}{\beta} \sqrt{\frac{q}{\sigma}}\right)^\mu}{I_\mu\left(\frac{2}{\beta} \sqrt{\frac{q}{\sigma}}\right)} \right]^2. \quad (\text{B19})$$

After the rescaling $Z_V = z_1 / (\sigma \beta^2)$, this corresponds to the result (186) given in the text.

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